IN SEARCH OF 'THE JAPANESE THEOREM'

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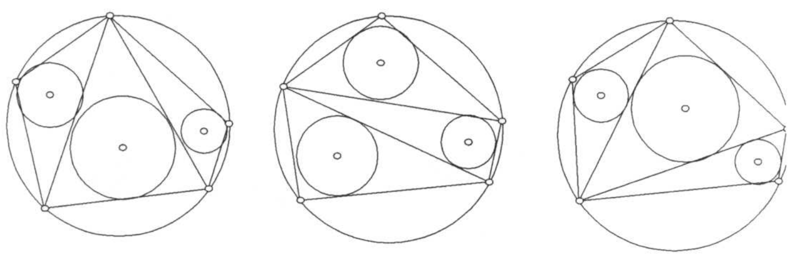
Note. The letters 'I', 'U', and 'M' refer to the first, second and third author. Our previous two papers show a variety of proofs of this theorem, seven in all, by Japanese mathematicians [1]. This paper relates the history and cultural setting of the theorem. It relies heavily on Prof. U's article [21].

Japanese Theorem. Triangulate a cyclic polygon by lines drawn from any vertex. The sum of the radii of the incircles of the triangles is independent of the vertex chosen.



Introduction. In 1996 a Masters Degree student at my university did her thesis on this theorem. It caught our attention when we saw an article in *The Mathematical Gazette,* where the theorem was stated without a proof [14]. After the theorem was proved [7], we began to wonder why it is called the 'Japanese Theorem'. Is the theorem perhaps very well-known in Japan? We made many inquiries both here and abroad. Finally, the answer came from Prof. Yoshida of Kyoto University. He sent a 15-page fax containing material in English, French, and Japanese. According to Prof. Yoshida the theorem is not so well-known in Japan [22].

The faxed material contained more interesting information. Around 1900 AD the quadrangle case (when the polygon is a quadrilateral) of the theorem had come to Japan from China. So, in Japan it was known as the 'Chinese Theorem'. The theorem soon attracted many Japanese mathematicians. It was inscribed on a wooden tablet (called *Sangaku* in Japanese) and displayed as an offering in a Shinto shrine. In 1905 a Japanese mathematician, Yoshio Mikami, generalized the theorem to the polygonal case.

This story heightened my curiosity even further. Here is a theorem, which is transported from one culture to another where it gets energized, enshrined and extended. How did its name change from the 'Chinese Theorem' to the 'Japanese Theorem'? Does it have some other names too? Is there perhaps another theorem with the name 'Japanese Theorem'? In this paper we will try to answer these questions. We will trace its origin and follow its trail through the cultures of China, Japan, and the western world. We will list the variety of names it assumed at different stages and some confusion in the literature.

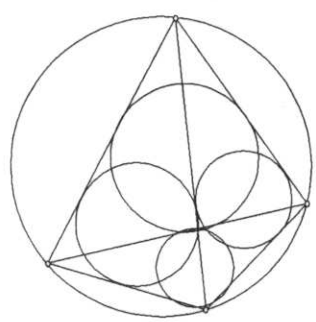
Kyoto Collaboration. In the summer of 1999, I visited the Kyoto University library. After signing the guest register I was permitted to use the facilities. Inside a large room I saw some hundred monitors blinking Japanese characters at me. I felt a little depressed because I had only a rudimentary knowledge of the Japanese language, and I wondered if it was a foolish idea to work on this project. But, like many others, I have realized that researchers all over the world belong to a certain kind of fellowship, and they help you just because you are a fellow researcher. This project is a testimony to that cooperation and fellowship. My coauthors came to my rescue. Prof. M and I collected material and made Xerox copies. Prof. U also collected, read, and analyzed the documents. The project moved at a rapid pace. By the summer of 2000, when I attended ICME-IX at Makuhari, Japan, Prof. U had already finished writing his article (in Japanese), which Prof. M translated for me. Much of this paper is based on Prof. U's article [21]. Without the help of these two friends the project would have come to a halt.

Historical Background. The years 1639–1854 were a period of isolation for Japan. In 1639 the ruling Tokugawa shogunate made a proclamation prohibiting the Japanese from traveling abroad, or having any contact with the foreigners and their culture. The isolation ended in 1854 when Commodore Perry forced Japan to open herself for trade with the western world. Because of the defeat at the hands of foreigners and the resulting discontent, the Tokugawa shogunate lost power and was replaced by the Meiji rule in 1867. The new rule brought an enormous number of social changes and a rapid rise in technology and industrialization. This modernization is known as the 'Meiji Restoration'.

The Japanese culture borrowed much from its neighbor China. Buddhist religion, early mathematics, a script for writing and even the tea ceremony, all came from China. The period of isolation provided a golden opportunity for the local arts, crafts, and mathematics to flourish. The native mathematics is known as *Wasan,* which literally means Japanese mathematics. As in Greece, mathematics spread not just among the elite, but among all segments of society. A tradition of solving problems emerged among the citizens. Many of them, after solving the problem, inscribed it on a beautifully decorated wooden tablet and offered it to their favorite deity at a Shinto shrine or a Buddhist temple [19]. This intertwining of religion and mathematics was common among Islamic and other cultures in early times. The mathematical tablets are known as *Sangaku* in Japanese. While many Sangakus have disappeared, even today there are over 800 tablets hanging in various religious places all over Japan. The description of many Sangakus is contained in an old Japanese book, *Zoku-Shinpeki-Sanpo* (ZSS) [3]. This book is important to us because it has the earliest reference to our theorem. The Sangaku of our theorem was on display in the Tsuruoka-San'nosha shrine in the Ushu area (at present Yamagata and Akita Prefectures), but it has since disappeared.

Sangaku of our Theorem. What does the Sangaku say? According to ZSS the Sangaku of our theorem has the following inscription.

Problem. Draw six lines in the circle and make four circles inscribed in three of the lines. If the diameter of the southern, eastern, and western circle is 1 sun, 2 suns, and 3 suns, respectively, how long is the diameter of the northern circle?



Answer. 4 suns.

Art. Add the diameter of the western circle to that of the eastern one and subtract that of the southern one from it, and you will get that of the northern one. End.

Kansei 12 Pupil of MARUYAMA Ryogen
Toto MARUYAMA Tetsugoro Ryokan

'Sun' is an old Japanese unit of length and is approximately 1.19 inches, '**Art**' refers to the art of solving the problem, '**Kansei 12**' is a date which corresponds to 1800 AD, and '**Toto**' is Tokyo. The tablet was offered by Ryokan Maruyama, a disciple of Ryogen Maruyama (1757–1816), a famous Wasan mathematician.

This inscription follows a pattern found on almost all Sangakus. First, the problem, sometimes accompanied by a picture, is posed as a question, like 'How long?' or 'How many?' Then follows the answer and the 'Art', or the recipe for getting the answer. There is no proof or explanation as to why this recipe works. Because of the name Maruyama on the tablet, the theorem was also known in Japan as Maruyama's Theorem.

Origin and Extension of the Theorem. The following sources indicate that the theorem originated in China.

(1) In [6] W. J. Greenstreet wrote, "*The following theorem was recently sent to a Japanese mathematician by one of his Chinese friends*". He also refers to Neuberg's article [18], in which Neuberg writes that a letter dated March 29, 1905 was sent by Mr. Mikami to Herrn A. Gutzmer which contained an interesting theorem, "*which was sent by a Chinese friend to his Japanese friend.*"

(2) In [15] Mikami wrote, "*The following proposition is one among others that were proposed by a certain Chinese mathematician to a friend of mine*". Again in [16] he wrote, "*This problem is simple, but has rich content. It comes from a certain Chinese mathematician.*" He added, "*It is not clear whether this theorem was proved in China, or how the Chinese mathematicians have dealt with this kind of problem.*"

While we are certain that the theorem came from China, the identities of the Chinese mathematician who sent the theorem, and the Japanese mathematician who received it are not known. This part of the puzzle remains unsolved. The theorem sent from China was for a cyclic quadrilateral, but it was extended to a polygon by Mikami in 1905 [15], and by Sawayama in 1906 [20].

Changing Names. Originally the name of the theorem was the 'Chinese Theorem' and later 'Maruyama's Theorem'. But, when the theorem found its way to the western world its name began to change. What do you call an unknown theorem that had come from Japan? Let us see how different authors chose its name. First, in 1906 Mr. Greenstreet wrote an article, 'Japanese Mathematics' in *The Mathematical Gazette,* where our theorem appeared without any name [6]. Next, we see the theorem in Roger Johnson's classic geometry textbook with the title, 'Theorem of Oriental Origin' [13]. Then, in a delightful series of books, *Mathematical Gems,* the author Honsberger calls our theorem, 'An Old Japanese Theorem' [12]. Finally, in 1993 Nick MacKinnon gives our theorem its present name, 'The Japanese Theorem' [14].

Another Theorem. Is there perhaps another theorem known as 'Japanese Theorem'? Surprisingly, the answer is 'Yes'. We will call this other theorem, 'Japanese Theorem-II', or 'JT-II'. This JT-II shows up at different places with slightly different wording. In [10] Prof. Hayashi writes that the following theorem appeared in 'Mathesis' (1896).

"Let the centers of circumcircle, incircle and three escribed circles of triangle ABC be O, I, I_a, I_b, I_c, and radius of each of these be R, r, r_a, r_b, r_c. And let the triangle made of the fourth common tangents to any two of the escribed circles be $A_1B_1C_1$. Then the center of the incircle of triangle $A_1B_1C_1$ is coincident with the center of the circumcircle of triangle $I_aI_bI_c$, and its radius r_1 is

$$r_1 = 2R + r = (r + r_a + r_b + r_c)/2."$$

Hayashi writes that this theorem is found in a Japanese book, 'Sanpo-Hengyo-Shinan' by Fukuda [5]. JT-II is also seen as Problem 2.5.4 on page 34 of [4]. Interestingly, JT-II was also displayed as a Sangaku. It was offered in April 1803 at the Niigata-Hachimangu shrine by Kinpei Hogo Yamamoto. Unfortunately this Sangaku too has disappeared, but we can see its description in another book on Sangaku by Nakamura [17].

There is some confusion because of the identical name for two different theorems. In Mathesis two authors have used the name 'Japanese Theorem', Herbiet in Mathesis Volume XL [11], and Erhart in Volume LX [2]. But Herbiet's theorem is the quadrangle case of our 'Japanese Theorem', whereas Erhart's theorem is JT-II.

Summary. A Chinese mathematician sent this theorem to a Japanese person, who was a friend of Yoshio Mikami. So in Japan its name was 'Chinese Theorem'. In 1800 Ryokan Maruyama inscribed this theorem on a Sangaku, so the theorem also got the name Maruyama's theorem. In 1905 Yoshio Mikami extended the theorem from a quadrangle to a polygon and reported it to the western world but still kept the name 'Chinese Theorem'. When the theorem reached the western world there was a gradual evolution in its name: no name (Greenstreet, 1906), 'Theorem of Oriental Origin' (Roger Johnson, 1929), 'An Old Japanese Theorem' (Ross Honsberger, 1985), and finally 'Japanese Theorem' (MacKinnon, 1993). To complicate matters even further, there is another theorem with the same name, 'Japanese Theorem' causing a bit of confusion in the literature because the name 'Japanese Theorem' is given to two different theorems in the same journal within a 25-year period.

Conclusion. Even though the theorem originated in China and had the name 'Chinese Theorem', the people working on it, extending it, and reporting it to the western world were all Japanese - Sawayama, Mikami, and Hayashi. There are numerous theorems which look similar to our theorem. In the epilogue of his article [21] Prof. U points out that between 1800 and 1850 Wasan mathematicians worked on many such theorems on circles and triangles. One can see some of these theorems in Chapter 2 of the Japanese Temple Geometry book [4]. All of them have the same 'flavor', but none has an exciting story as our Japanese theorem. Perhaps behind the name of every theorem there lies an interesting history waiting for some enterprising researcher to unfold. For me this was much more than a Math project. The agony of trying to read Japanese, but also the pleasure of two highly competent and helpful researchers made it an exciting and unforgettable adventure.

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