A NOTE ON CLOSED FUNCTIONS

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Recently, while preparing a lecture on closed functions for a general topology course, what appear to be new characterizations of such functions were discovered. The purpose of this note is to present these characterizations. Recall, if X, Y are topological spaces, a function $f\colon X \to Y$ is a *closed function* if $f(A)$ is a closed subset of Y whenever A is a closed subset of X. In the sequel, topological spaces will be referred to as simply spaces. If A is a subset of a space, $\mathrm{cl}(A)$ will denote the closure of A, $d(A)$ will be the set of limit points of A, and $\Sigma(A)$ ($\Sigma(x)$ if $A = \{x\}$) will denote the collection of open subsets which contain A. The following equivalent statements for spaces X, Y, and $f\colon X \to Y$ appear in a number of books on general topology [2].

1) The function f is a closed function.
2) The function f satisfies $\mathrm{cl}(f(A)) \subset f(\mathrm{cl}(A))$ for each $A \subset X$.
3) For each $y \in Y$ and $W \in \Sigma(f^{-1}(y))$, some $H \in \Sigma(y)$ satisfies $f^{-1}(H) \subset W$.

It is of significant pedagogical value, when beginning the study of closed functions in elementary general topology courses, to give classes of continuous functions, which have been previously encountered by the students, which provide examples of closed functions and examples of functions which are not closed.

Example 1. A complex polynomial is a closed function.

Verification. If the polynomial P is constant, it is a closed function. If $n \geq 1$ and $P(z) = \sum_{k=0}^{n} a_k z^{n-k}$ is a complex polynomial of degree n, it is easy to use the inequality $|z^n - z_0^n| \leq (|z - z_0| + |z_0|)^n - |z_0|^n$, which was established in [1], to prove that P is a continuous function. Also, it is not difficult to see that if $|z|$ is taken arbitrarily large, then $|P(z)|$ becomes arbitrarily large. Now, let A be a closed subset of the complex plane and let z_n be a sequence in A such that $P(z_n) \to w$. Then $P(z_n)$ is a bounded sequence. Hence, z_n is a bounded sequence in A. By the Bolzano-Weierstrass theorem, there is a subsequence, called again z_n such that $z_n \to x$, and $x \in A$ since A is closed. By the continuity of P, it follows that $P(z_n) \to P(x)$ and $P(x) = w$.

Example 2. A real-valued continuous periodic function with a non-discrete subset of the reals as its domain is not a closed function.

Verification. Let f be the function with period p, let x be in the domain of f, and let x_n be a sequence of distinct elements of the domain such that $x_n \to x$. Let $A = \{x_n + np : n = 1, \ldots\}$. Without loss, assume that $f(x_n) \neq f(x)$. Then $d(A) = \emptyset$, so A is a closed subset of the reals; and $f(x_n + np) = f(x_n) \to f(x)$. Hence, $f(A)$ is not a closed subset of the reals.

Example 2 shows that the circular functions are not closed functions.

Now, to the main results of the note.

Theorem 1. Let X, Y be spaces. A function $f: X \to Y$ is a closed function if and only if $d(f(A)) \subset f(d(A))$ for each $A \subset X$.

Proof. If $d(f(A)) \subset f(d(A))$ for each $A \subset X$, and A is a closed subset of X, then $\mathrm{cl}(f(A)) = f(A) \cup d(f(A)) \subset f(A) \cup f(d(A)) \subset f(A)$. For the converse, let $f: X \to Y$ be a closed function and let $A \subset X$. Let $y \in d(f(A))$. Then $y \in \mathrm{cl}(f(A) - \{y\}) = \mathrm{cl}(f(A - f^{-1}(y))) \subset f(\mathrm{cl}(A - f^{-1}(y)))$. Hence, $f^{-1}(y) \cap \mathrm{cl}(A - f^{-1}(y)) \neq \emptyset$. Choose $x \in f^{-1}(y) \cap \mathrm{cl}(A - f^{-1}(y))$. Then $x \in d(A)$ and $y = f(x) \in f(d(A))$. The proof is complete.

A space is called *Bolzano-Weierstrass* if every countably infinite subset of the space has a limit point. Corollary 1 is a consequence of Theorem 1.

Corollary 1. Let X, Y be spaces with Y Bolzano-Weierstrass, and let $f: X \to Y$ be a closed function with $f^{-1}(y)$ Bolzano-Weierstrass for each $y \in Y$. Then X is Bolzano-Weierstrass.

Proof. Let A be a countably infinite subset of X. If $f(A)$ is finite, then $A \cap f^{-1}(y)$ is infinite for some $y \in Y$. Hence, $d(A) \neq \emptyset$. If $f(A)$ is infinite, then $d(f(A)) \neq \emptyset$, so $d(A) \neq \emptyset$ follows from Theorem 1. The proof is complete.

Theorem 2. Let X and Y be spaces. Then $g: X \to Y$ is a closed function if and only if $g(X)$ is closed in Y and $g(V) - g(X - V)$ is open in $g(X)$ whenever V is open in X.

Proof. Suppose $g: X \to Y$ is a closed function. Clearly, $g(X)$ is closed in Y and $g(V) - g(X - V) = g(X) - g(X - V)$ is open in $g(X)$ when V is open in X. On the other hand, suppose $g(X)$ is closed in Y, $g(V) - g(X - V)$ is open in $g(X)$ when V is open in X, and let C be closed in X. Then $g(C) = g(X) - (g(X - C) - g(C))$ is closed in $g(X)$ and hence, closed in Y. The proof is complete.

Corollary 2. Let X and Y be spaces. Then a surjection $g: X \to Y$ is a closed function if and only if $g(V) - g(X - V)$ is open in Y whenever V is open in X.

Corollary 3. Let X and Y be spaces and let $g\colon X \to Y$ be a continuous closed surjection. Then the topology on Y is $\{g(V) - g(X - V) : V \text{ open in } X\}$.

Proof. Let W be open in Y. Then $g^{-1}(W)$ is open in X, and $g(g^{-1}(W)) - g(X - g^{-1}(W)) = W$. Hence, all open sets in Y are of the form $g(V) - g(X - V)$, V open in X. On the other hand, all sets of the form $g(V) - g(X - V)$, V open in X, are open in Y from Corollary 2. The proof is complete.

Theorem 3. Let X and Y be spaces with X normal and let $g\colon X \to Y$ be a continous closed surjection. Then Y is normal.

Proof. Let K, M be closed disjoint subsets of Y. Then $g^{-1}(K)$, $g^{-1}(M)$ are closed disjoint subsets of X. Let $V \in \Sigma(g^{-1}(K))$, $W \in \Sigma(g^{-1}(M))$ be disjoint. Then, $K \subset g(V) - g(X - V)$ and $M \subset g(W) - g(X - W)$. Further, by Corollary 2, $g(V) - g(X - V)$ and $g(W) - g(X - W)$ are open in Y, and clearly, $(g(V) - g(X - V)) \cap (g(W) - g(X - W)) = \emptyset$. The proof is complete.

References

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2. R. Engelking, *General Topology,* PWN – Polish Scientific Publishers, Warsaw, 1977.

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