SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

149. [2004, 129] *Proposed by Joe Howard, Portales, New Mexico and Les Reid, Southwest Missouri State University, Springfield, Missouri.*

Let A, B, C be the angles of a triangle. Show

$$\begin{aligned} 3+\cos A+\cos B+\cos C &\geq 2(\sin A\sin B+\sin B\sin C+\sin C\sin A) \\ &\geq 9(\cos A+\cos B+\cos C-1) \end{aligned}$$

with equality if and only if the triangle is equilateral.

Solution by the proposers. Let r, R, and s denote the inradius, circumradius, and semiperimeter, respectively. Euler's inequality ($R \geq 2r$) is well known. The following can be found in *Crux Mathematicorum,* [2029] 22 (3) (1996), p. 130 and on pages 45 and 55–56 of D. S. Mitrinović, J. E. Pečurić, and V. Volenec, *Recent Advances in Geometric Inequalities,* Kluwer Academic Pub., 1989.

$$\sin A\sin B+\sin B\sin C+\sin C\sin A=\frac{s^2+4Rr+r^2}{4R^2} \tag{1}$$

$$\cos A+\cos B+\cos C=1+\frac{r}{R} \tag{2}$$

$$4R^2+4Rr+3r^2\geq s^2\geq 16Rr-5r^2 \tag{3}$$

Using (1) and (2), the given inequality is equivalent to

$$4+\frac{r}{R}\geq 2\left(\frac{s^2+4Rr+r^2}{4R^2}\right)\geq 9\left(\frac{r}{R}\right)$$

which is equivalent to

$$8R^2-2Rr-r^2\geq s^2\geq 14Rr-r^2.$$

For the first inequality: from Euler's inequality ($R \geq 2r$); $R^2 \geq 4r^2$ and $3R^2 \geq 6Rr$ which implies $4R^2 \geq 6Rr + 4r^2$. By this and (3),

$$8R^2 - 2Rr - r^2 \geq 4R^2 + 4Rr + 3r^2 \geq s^2.$$

For the second inequality: from Euler's inequality ($R \geq 2r$); $2Rr \geq 4r^2$ and with (3) implies

$$s^2 \geq 16Rr - 5r^2 \geq 14Rr - r^2.$$

The equality is clear when the triangle is equilateral. Conversely, if $8R^2 - 2Rr - r^2 = s^2$, then the first part of (3) yields

$$4R^2 + 4Rr + 3r^2 \geq 8R^2 - 2Rr - r^2$$

or

$$0 \geq 4R^2 - 6Rr - 4r^2 = 2(R - 2r)(2R + r).$$

This implies that $2r \geq R$, which along with Euler's inequality forces $2r = R$, hence the triangle must be equilateral. A similar argument works for the other inequality.

Also solved by Scott H. Brown, Auburn University, Montgomery, Alabama and Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan.

150. [2004, 130] *Proposed by Ovidui Furdui, Western Michigan University, Kalamazoo, Michigan.*

Evaluate

$$\iint_D \left\{\frac{1}{x+y}\right\} dx\, dy,$$

where $D = [0,1] \times [0,1]$ and $\{a\}$ is the fractional part of a.

Solution by Huizeng Qin, Shandong University of Technology, Zibo, People's Republic of China. Let

$$x = \frac{1}{2}(u - v), \qquad y = \frac{1}{2}(u + v). \tag{1}$$

Then,

$$\iint_D \left\{\frac{1}{x+y}\right\} dx\, dy = \frac{1}{2}\left(\iint_{D_1} + \iint_{D_2}\right)\left\{\frac{1}{u}\right\} du\, dv = I_1 + I_2, \tag{2}$$

where

$$D_1 = \{-u \le v \le u,\ 0 \le u \le 1\} \text{ and } D_2 = \{u-2 \le v \le 2-u,\ 1 \le u \le 2\}.$$

Now, we evaluate I_1 and I_2, separately. Since $\{1/u\} = 1/u$ when $u > 1$, we first have

$$I_2 = \frac{1}{2}\iint_{D_2} \frac{1}{u} du\, dv = \frac{1}{2}\int_1^2 \int_{u-2}^{2-u} \frac{1}{u} dv\, du = \int_1^2 \frac{2-u}{u}\, du = 2\ln 2 - 1. \tag{3}$$

To evaluate I_1, we rewrite $\{a\}$ as $a - \lfloor a \rfloor$, where $\lfloor a \rfloor$ is the integer part of a. Now, we can use the well known result

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{4}$$

to obtain

$$\begin{aligned} I_1 &= \frac{1}{2}\int_0^1 \int_{-u}^{u} \left(\frac{1}{u} - \left\lfloor \frac{1}{u} \right\rfloor\right) dv\, du = 1 - \int_0^1 u \left\lfloor \frac{1}{u} \right\rfloor du \qquad (5) \\ &= 1 - \int_1^{\infty} \frac{\lfloor v \rfloor}{v^3} dv = \sum_{n=1}^{\infty} \int_n^{n+1} \frac{n}{v^3} dv \\ &= 1 - \frac{1}{2}\sum_{n=1}^{\infty} n\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) \\ &= 1 - \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{(n+1)^2}\right) \\ &= \frac{1}{2} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \\ &= \frac{1}{2} - \frac{1}{2}\left(\frac{\pi^2}{6} - 1\right) = 1 - \frac{\pi^2}{12}. \end{aligned}$$

Combining (2), (3), and (5), we get

$$\iint_D \left\{\frac{1}{x+y}\right\} dx\,dy = 2\ln 2 - \frac{\pi^2}{12}.$$

Also solved by the proposer.

151. [2004, 130] *Proposed by José Luis Diaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain.*

Solve the differential equation

$$\sqrt{1+y^2}\,x^2 e^{\arctan x} dx + \sqrt{1+x^2}\,dy = 0.$$

Solution by the proposer. Dividing both sides of the given equation by $\sqrt{(1+x^2)(1+y^2)}$ and rearranging terms yields

$$-\frac{dy}{\sqrt{1+y^2}} = \frac{x^2 e^{\arctan x}}{\sqrt{1+x^2}} dx. \tag{1}$$

Integrating both sides of (1), we get

$$-\ln(y+\sqrt{1+y^2}) = -\sinh^{-1}(y) = F(x),$$

where

$$F(x) = \int \frac{x^2 e^{\arctan x}}{\sqrt{1+x^2}} dx.$$

To obtain $F(x)$, we call

$$I_1 = \int \frac{e^{\arctan x}}{\sqrt{1+x^2}} dx$$

and

$$I_2 = \int \frac{xe^{\arctan x}}{\sqrt{1+x^2}} dx.$$

Then,

$$I_1 = \int \frac{e^{\arctan x}}{\sqrt{1+x^2}} dx = \int \frac{\sqrt{1+x^2} e^{\arctan x}}{1+x^2} dx$$

$$= \int \sqrt{1+x^2} d(e^{\arctan x}) = \sqrt{1+x^2} e^{\arctan x} - I_2$$

and so,

$$I_1 + I_2 = \sqrt{1+x^2} e^{\arctan x}. \tag{2}$$

On the other hand,

$$I_2 = \int \frac{xe^{\arctan x}}{\sqrt{1+x^2}} dx = \int \frac{x\sqrt{1+x^2} e^{\arctan x}}{1+x^2} dx$$

$$= \int x\sqrt{1+x^2} d(e^{\arctan x})$$

$$= x\sqrt{1+x^2} e^{\arctan x} - I_1 - 2 \int \frac{x^2 e^{\arctan x}}{\sqrt{1+x^2}} dx.$$

That is,

$$F(x) = \frac{x\sqrt{1+x^2} e^{\arctan x} - (I_1 + I_2)}{2}.$$

Taking into account (2), we obtain

$$F(x) = \frac{1}{2}(x-1) e^{\arctan x} \sqrt{1+x^2}.$$

Therefore,

$$\sinh^{-1}(y) = \frac{1}{2}(1-x)e^{\arctan x}\sqrt{1+x^2} + C$$

or

$$y = \sinh\left(\frac{(1-x)e^{\arctan x}\sqrt{1+x^2}}{2} + C\right), \ C \in \mathbb{R}$$

and we are done.

Also solved by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan and Kenneth Davenport, Dallas, Pennsylvania.

152. [2004, 130] *Proposed by Joe Flowers and Doug Martin (student), Texas Lutheran University, Seguin, Texas.*

Let

$$F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

denote the Laplace transform of $f(t)$. Find $L[\sin^n bt]$, where b is any real constant and n is any non-negative integer.

Solution by the proposers. Let

$$F_n(s) = L[\sin^n bt].$$

The results

$$F_0(s) = \frac{1}{s} \quad \text{and} \quad F_1(s) = \frac{b}{s^2+b^2}$$

are well-known. For $n > 1$, let

$$f(t) = \sin^n bt.$$

Then,

$$f'(t) = nb \cdot \cos bt \cdot \sin^{n-1} bt$$

and

$$f''(t) = nb^2\Big((n-1)\sin^{n-2} bt - n \cdot \sin^n bt\Big)$$

and therefore,

$$L[f''(t)] = b^2\Big(n(n-1)F_{n-2}(s) - n^2 F_n(s)\Big).$$

Also, from the theory of Laplace transforms,

$$L[f''(t)] = s^2 F_n(s) - s \cdot f(0) - f'(0) = s^2 F_n(s).$$

From these we obtain the second degree recurrence formula

$$F_n(s) = \frac{b^2 \cdot n(n-1)}{s^2 + n^2 b^2} \cdot F_{n-2}(s)$$

and then by iteration,

$$F_n(s) = \frac{b^n \cdot n!}{s \cdot \prod_{k=1}^{n/2} (s^2 + 4k^2 b^2)}$$

for even values of n and

$$F_n(s) = \frac{b^n \cdot n!}{\prod_{k=0}^{(n-1)/2} (s^2 + (2k+1)^2 b^2)}$$

for odd values of n.

Also solved by Thomas Dence, Ashland University, Ashland, Ohio and Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan. A partial solution was also received.