ADDING SOMETHING EXTRA

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It requires more innate intellectual capacity to dispose of this apparently childish thing than to grasp the theory of relativity.

E. T. Bell

1. Introduction. The "thing" referred to in the above quote is the Diophantine equation

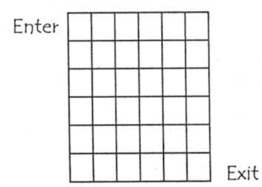
$$y^3 = x^2 + 2$$

and to "dispose of" this thing is to find all its positive integer solutions [1]. It is clear that $x = 5$ and $y = 3$ is a solution, but Fermat, as a challenge to other mathematicians, asked for a proof that this is the *only* such solution [5]. The challenge was met, more than a hundred years later, by Euler. Euler's short proof is easily followed by students taking a course in algebra or number theory. So what would lead E. T. Bell to make such a bold claim for the proof?

Euler's proof is remarkable because it uses *complex numbers* to prove something about *positive integers.* In other words, extra structure is added to help unravel the secret of the equation. This principle of "adding something extra" in a proof is discussed by George Polya in his classic book *How to Solve It* [4]. So how does adding extra structure help us solve a problem? In this note, we illustrate how this principle works, then give a sketch of Euler's proof in light of this principle. In our classes, we have found that explicitly identifying when extra structure is added in a proof gives students an appreciation for the ingenuity required to invent the proof, as well as an understanding of a principle they can recognize when they encounter its use in other proofs.

First, to see how adding something extra can help solve a problem, we consider a simple problem where this principle is clearly and strikingly used.

2. The Maddening Mouse Maze. A maze is in the shape of a square with six rooms to a side. Each room is connected by a door to each of its contiguous neighbors. The entrance and exit are at diagonally opposite corners, as shown in the figure below. A mouse entering the maze must pass through each room exactly once in order for the exit door to open. What route should the mouse take?



After several attempts the mouse might wonder whether such a route is possible. But how could the mouse give a convincing argument that no such route exists? Here is a proof: Color the rooms alternately red and black in a checkerboard pattern. Observe that any route through the maze necessarily alternates between red and black rooms. Since the maze has an even number of rooms, the required path must begin and end in rooms of opposite color. But the entrance and exit are in rooms of the same color, so traversing this maze is impossible!

The extra element: This short proof utilizes aspects entirely outside the original problem. Indeed, coloring the rooms is not relevant – the maze either can or cannot be traversed irrespective of the colors of the rooms. But adding color in the above fashion is the extra element that allows us to give a convincing argument that traversing the maze is impossible.

3. Back to Euler. Euler contemplated Fermat's equation

$$y^3 = x^2 + 2 \tag{1}$$

for many years. His eventual solution reaches far outside the realm of the positive integers. Euler factored the right hand side to get the equation

$$y^3 = (x + \sqrt{-2})(x - \sqrt{-2}). \tag{2}$$

This led him to look at the original Diophantine equation in a whole new setting: What are the solutions in "algebraic integers" of the form $m + n\sqrt{-2}$ [2]? (By analogy with the Maddening Mouse Maze, we can think of $m+n\sqrt{-2}$ as a "colored" version of the integer m). Of course every ordinary integer is one of these algebraic integers, so finding the solutions in algebraic integers would also solve the original problem. Here is how the added structure helps us find all the solutions of (1).

We begin by considering the collection

$$R = \{m + n\sqrt{-2} \ : \ m, n \in \mathbb{Z}\}.$$

Now R is a ring that has many of the same properties as the ring $\mathbb{Z}$. For example, like $\mathbb{Z}$, the ring R has *prime* elements, that is, elements that cannot be factored (excluding 1 and -1 as factors). And, like $\mathbb{Z}$, the ring R has the crucial property of *unique factorization* - that is, each element can be factored in an essentially unique way (up to the order and sign of the factors) into prime elements of R [3]. For example, in R we have $1 + 4\sqrt{-2} = (3 + \sqrt{-2})(1 + \sqrt{-2})$ and each of these factors is prime.

Now, it is not too difficult to show that $x + \sqrt{-2}$ and $x - \sqrt{-2}$ have no factor in common in R. It follows from (2) and unique factorization that each of these elements of R must itself be a cube. So, there are ordinary integers m and n such that

$$(m + n\sqrt{-2})^3 = x + \sqrt{-2}.$$

Expanding the left-hand side we get

$$m^3 - 6mn^2 + (3m^2n - 2n^3)\sqrt{-2} = x + \sqrt{-2}.$$

Comparing coefficients of $\sqrt{-2}$ we get $n(3m^2 - 2n^2) = 1$. So we must have $n = \pm 1$ and $3m^2 - 2n^2 = \pm 1$, from which we get $m = \pm 1$. Now equating the real parts gives $x = m^3 - 6mn^2$, and so we must have $x = \pm 5$. Thus, $x = 5$ and $y = 3$ is the only positive integer solution.

The extra element: Euler's solution adds a whole extra dimension to the problem. He frees it from the confines of the positive integers and recasts it in the setting of the ring of algebraic integers of the form $m + n\sqrt{-2}$. To begin with we were not interested in solutions that contain $\sqrt{-2}$, but somehow this additional structure provides the setting that unveils the mystery!

4. Conclusion. The principle of adding new structure to solve a problem characterizes many of the great discoveries in mathematics. But the principle can be applied to solve many ordinary problems. A common example in calculus courses is evaluating an integral by trigonometric substitution – even though there are no trigonometric functions in the integrand, substituting a trigonometric function for the variable is the extra element needed to unravel the integral. It is interesting and instructive to search for instances in mathematics where this principle is used.

References

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