SOME REMARKS ON THE SUM OF AN OLD SERIES

Rasul A. Khan

In this note we use some combinatorial identities to derive a formula for the sum of the series

$$S(p) = \sum_{n=0}^{\infty} (m+n+1)^p \binom{n+m}{m} x^n, \quad |x| < 1,$$

in the form $P(x)/(1-x)^{m+p+1}$, where $P(x)$ is a polynomial of degree $p-1$ with known coefficients a_j, $0 \le j \le p-1$. When specialized for $m = 0$, the resulting sum gives a formula for

$$\sum_{n=1}^{\infty} n^p x^n \quad (|x| < 1).$$

The general formula also provides an alternative method for determining the moments of a negative binomial distribution. Conversely, the negative binomial distribution can be used to find a recursive formula for the sum of the above series $S(p)$.

1. A Combinatorial Identity. In what follows we write

$$x_{(r)} = x(x-1)(x-2)\cdots(x-r+1)$$

for any real x and positive integer r, and in particular $x_{(r)} = x!/(x-r)!$ if x is also a positive integer, and $x_{(0)} = 1$, etc. Also, for a function $\psi(t)$, $\psi^{(k)}(a)$ denotes the kth derivative evaluated at a. The identities in Lemma 1 can be found in disguised forms in Feller [2]. These identities in turn imply the main identity in Lemma 2. A generating function method is used for their proofs for the sake of completeness.

Lemma 1. For any real number m and positive integers n and r we have

$$\text{(a)}\quad \sum_{j=0}^{n}(-1)^j(m-j)^r\binom{n}{j} = \begin{cases} 0, & \text{for all } r<n, \\ n!, & \text{for } r=n. \end{cases}$$

$$\text{(b)}\quad \sum_{j=0}^{n}(-1)^j\binom{n}{j}(n+m-j)_{(r)} = \begin{cases} 0, & \text{for all } r<n, \\ n!, & \text{for } r=n. \end{cases}$$

Proof. Let

$$g(t) = e^{mt}(1-e^{-t})^n = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\exp((m-j)t).$$

Since $g^{(r)}(0)=0$ for all $r<n$, and $g^{(n)}(0)=n!$, (a) follows. Now let

$$g_1(t) = t^m(t-1)^n = \sum_{j=0}^{n}(-1)^j\binom{n}{j}t^{m+n-j}.$$

Since $g_1^{(r)}(1)=0$ for all $r<n$, and $g^{(n)}(1)=n!$, (b) follows.

To state the second lemma we define the following sequence a_n for positive integers m and p. This sequence is needed to find the sum of the series $S(p)$. The sequence a_n is defined as:

$$a_n = \sum_{i=0}^{n}(-1)^i(m+n+1-i)^p\binom{m+p+1}{i}\binom{n+m-i}{m}. \tag{1}$$

Lemma 2.

$$\sum_{j=0}^{n}(-1)^j(m+n+1-j)^p\binom{n}{j}(m+n-j)_{(r)} = 0,\ r=n-p-1.$$

Moreover, $a_n = 0$ for all $n \geq p$, where a_n is defined by (1).

Proof. Let $g(t) = t^m \exp((m+1)(t-1))(te^{t-1}-1)^n$. Clearly,

$$g(t) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} t^{m+n-j} \exp((m+n+1-j)(t-1)). \tag{2}$$

Let $f_1(t) = t^{m+n-j}$ and $f_2(t) = \exp((m+n+1-j)(t-1))$. Obviously,

$$f_1^{(r)}(1) = (m+n-j)_{(r)},\ r = n-p-1, \text{ and } f_2^{(p)}(1) = (m+n+1-j)^p.$$

It is easy to verify that $g^{(k)}(1) = 0$ for all $k < n$. By repeated differentiation of (2) and repeated use of Lemma 1 we obtain

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} f_1^{(\alpha)}(1) f_2^{(\beta)}(1) = 0,\ \alpha+\beta = k < n. \tag{3}$$

In particular, (3) holds if $\alpha = n-p-1$ and $\beta = p$, and the identity is proved. To prove the noted property of a_n we consider two separate cases. If $n = p$, we have

$$(m+p+1-i)^p \binom{m+p+1}{i} \binom{p+m-i}{m} = \frac{(m+p+1)!}{m!p!} (m+p+1-i)^{p-1} \binom{p}{i},$$

and $a_p = 0$ by Lemma 1(a). If $n > p$ (i.e. $n \geq p+1$) we write

$$\binom{m+p+1}{i} \binom{n+m-i}{m} = \frac{(m+p+1)!}{m!n!} \binom{n}{i} (n+m-i)_{(r)}, \quad r = n-p-1.$$

Then, "showing $a_n = 0$ for all $n > p$" is equivalent to the main identity in Lemma 2. Hence, $a_n = 0$ for all $n \geq p$.

2. An Application. The mth derivative of the geometric series

$$\sum_{n=0}^{\infty} x^n = (1-x)^{-1}, \quad |x| < 1,$$

gives the well-known sum [4]

$$\sum_{n=0}^{\infty} \binom{n+m}{m} x^n = \frac{1}{(1-x)^{m+1}}, \quad |x| < 1. \tag{4}$$

If $0 < x < 1$, then

$$f(n; m, x) = \binom{n+m}{m} (1-x)^{m+1} x^n, \quad n = 0, 1, 2, \ldots$$

is the probability function of a negative binomial distribution, and (4) can be interpreted as

$$\sum_{n=0}^{\infty} f(n; m, x) = 1 \quad (\text{cf. [2]}).$$

Motivated by this observation and the related moments problem we seek to derive a formula for

$$\sum_{n=0}^{\infty} (m+n+1)^p \binom{n+m}{m} x^n, \quad |x| < 1,$$

where m (≥ 0) and p (≥ 0) are integers. It is claimed that

$$\sum_{n=0}^{\infty} (m+n+1)^p \binom{n+m}{m} x^n = \frac{P(x)}{(1-x)^{m+p+1}}, \quad |x| < 1, \tag{5}$$

where

$$P(x) = \sum_{j=0}^{p-1} a_j x^j$$

with known coefficients a_j $(0 \le j \le p-1)$, and $P(x) = 1$ if $p = 0$. To see this, let

$$b_n = (m+n+1)^p \binom{n+m}{m}, \quad c_n = (-1)^n \binom{m+p+1}{n},$$

and consider the power series

$$A = \sum_{n=0}^{\infty} b_n x^n, \quad \text{and} \quad B = (1-x)^{m+p+1} = \sum_{n=0}^{m+p+1} c_n x^n.$$

Then

$$AB = \sum_{n=0}^{\infty} a_n x^n$$

by the Cauchy product formula, and for $n = 0, 1, 2, \dots$, the coefficients a_n are given by (1). Since $a_n = 0$ for all $n \ge p$ by Lemma 2,

$$AB = \sum_{n=0}^{p-1} a_n x^n = P(x), \quad \text{and} \quad A = P(x)/B,$$

and (5) follows.

Special Case. Put $m = 0$ in (5), and the formula becomes

$$\sum_{n=0}^{\infty} (n+1)^p x^n = \sum_{n=1}^{\infty} n^p x^{n-1} = \frac{P(x)}{(1-x)^{p+1}}, \tag{6}$$

where

$$P(x) = \sum_{j=0}^{p-1} a_j x^j,$$

and a_n in (1) reduces to

$$a_n = \sum_{i=0}^{n} (-1)^i (n+1-i)^p \binom{p+1}{i}.$$

Clearly, $a_0 = 1$ and $a_n = 0$ for all $n \geq p$. In this special case there is one more interesting observation, namely, $a_{p-1} = 1$ for all $p \geq 1$. This can be seen by noting that

$$a_{p-1} = \sum_{i=0}^{p-1} (-1)^i (p-i)^p \binom{p+1}{i} = \sum_{i=0}^{p+1} (-1)^i (p-i)^p \binom{p+1}{i} - (-1)^{p+1}(-1)^p$$

$$= \sum_{i=0}^{p+1} (-1)^i (p-i)^p \binom{p+1}{i} + 1.$$

The last sum is clearly zero by Lemma 1(a), and hence, $a_{p-1} = 1$ for all $p \geq 1$. Obviously, (6) gives a formula for

$$\sum_{n=1}^{\infty} n^p x^n$$

as

$$\frac{xP(x)}{(1-x)^{p+1}},$$

which was also noted by Clarke [1] and Stalley [4]. Another curious observation is worth mentioning. Since $P(x)$ is a polynomial of degree $p-1$ and the a_j's are integers,

$$S_p = \sum_{n=1}^{\infty} \frac{n^p}{2^n} = 2^p P\left(\frac{1}{2}\right)$$

is always an even integer. It follows from (5) that

$$S_p(m) = \sum_{n=1}^{\infty} (n+m)^p \binom{n+m-1}{m} \frac{1}{2^n} = 2^{p+m} P\left(\frac{1}{2}\right)$$

is also an even integer for any $m \geq 0$ and $p \geq 0$. Clearly, $S_p = 2^p + 2^{p-1}a_1 + 2^{p-2}a_2 + \cdots + 2$. In this special case one can easily check that $S_1 = 2$, $p = 2$, $a_1 = 1$, $S_2 = 6$, $p = 3$, $a_1 = 4$, $a_2 = 1$, $S_3 = 26$, $p = 4$, $a_1 = 11$, $a_2 = 11$, $a_3 = 1$, $S_4 = 150$, and $p = 5$, $a_1 = 26$, $a_2 = 66$, $a_3 = 26$, $a_5 = 1$, $S_5 = 1082$, etc.

3. A Probabilistic Approach and a Recursive Formula for S(p). Let $0 < x < 1$, and

$$P(\nu = n) = f(n; m, x) = \binom{n+m}{m} (1-x)^{m+1} x^n, \quad n = 0, 1, 2, \ldots$$

be the probability function of a negative binomial distribution (cf. [2]). Let $X = \nu + m + 1$. Then the pth moment of X can be obtained from (5), which is given by

$$EX^p = E(\nu + m + 1)^p = \sum_{n=0}^{\infty} (n+m+1)^p P(\nu = n) = \frac{P(x)}{(1-x)^p}. \tag{7}$$

In particular, if $p = 1$, then $a_0 = m + 1$, and $EX = (m+1)/(1-x)$, and if $p = 2$, then $a_0 = (m+1)^2$, $a_1 = m + 1$, and

$$EX^2 = \frac{(m+1)^2 + (m+1)x}{(1-x)^2}.$$

Moreover, it is obvious that the variance $\sigma^2 = E(X - EX)^2 = EX^2 - (EX)^2) = (m+1)x/(1-x)^2$. These formulas are known in the literature where it is customary to use q for x and p for $1 - x$, and $r = m + 1$. Of course, any moment of X can be calculated from (7).

There is an alternative derivation of the sum

$$S(p) = \sum_{n=0}^{\infty} (n+m+1)^p \binom{n+m}{m} x^n$$

by calculus via the negative binomial distribution. This interesting interplay between $S(p)$ and the negative binomial distribution is now discussed. This method produces a recursive formula for $S(p)$. For simplicity we write $r = m+1$ (m being a positive integer) in the above negative binomial distribution. Thus,

$$f(n; r, x) = (1-x)^r \binom{r+n-1}{r-1} x^n, \quad n = 0, 1, 2, \dots ,$$

and the associated moment generating function $\phi^*(\theta) = E\exp(\theta X)$ is given by

$$\phi^*(\theta) = \sum_{n=0}^{\infty} e^{n\theta} f(n; r, x) = \frac{(1-x)^r}{(1-xe^\theta)^r} = \frac{1}{(1-\beta(e^\theta - 1))^r}, \quad \theta < -\ln x,$$

where $\beta = x/(1-x)$. Consequently,

$$\phi(\theta) = E\exp(\theta(X+r)) = \exp(r\theta)/(1-\beta(e^\theta-1))^r,$$

and $\phi'(\theta) = r(1+\beta)\phi(\theta)/(1-\beta(e^\theta-1))$. Thus,

$$\phi'(\theta)(1-\beta(e^\theta-1)) = r(1+\beta)\phi(\theta). \tag{8}$$

Differentiating (8) $p-1$ ($p \geq 1$) times, by Leibniz's formula we have

$$\sum_{j=0}^{p-1} \binom{p-1}{j} \phi^{(j+1)}(\theta)(1-\beta(e^\theta-1))^{(p-1-j)} = r(1+\beta)\phi^{(p-1)}(\theta),$$

which evaluated at $\theta = 0$ gives

$$-\beta \sum_{j=0}^{p-2} \binom{p-1}{j} \phi^{(j+1)}(0) + \phi^{(p)}(0) = r(1+\beta)\phi^{(p-1)}(0).$$

This can be written as

$$-\beta \sum_{j=1}^{p-1} \binom{p-1}{j-1} \phi^{(j)}(0) + \phi^{(p)}(0) = r(1+\beta)\phi^{(p-1)}(0).$$

Since

$$\binom{p-1}{j-1} = \frac{j}{p}\binom{p}{j},$$

we obtain

$$\begin{aligned}
\phi^{(p)}(0) &= \beta \sum_{j=1}^{p-1} \frac{j}{p}\binom{p}{j}\phi^{(j)}(0) + r(1+\beta)\phi^{(p-1)}(0) \\
&= \beta \sum_{j=1}^{p-1} \frac{p-p+j}{p}\binom{p}{j}\phi^{(j)}(0) + r(1+\beta)\phi^{(p-1)}(0) \\
&= \beta \sum_{j=1}^{p-1} \binom{p}{j}\phi^{(j)}(0) - \beta \sum_{j=1}^{p-1} \frac{p-j}{p}\binom{p}{j}\phi^{(j)}(0) + r(1+\beta)\phi^{(p-1)}(0).
\end{aligned}$$

Since

$$\frac{p-j}{p}\binom{p}{j} = \binom{p-1}{j},$$

we have

$$\phi^{(p)}(0) = \beta \sum_{j=1}^{p-1} \binom{p}{j} \phi^{(j)}(0) - \beta \sum_{j=1}^{p-1} \binom{p-1}{j} \phi^{(j)}(0) + r(1+\beta)\phi^{(p-1)}(0). \tag{9}$$

This recursive formula can be used for computing moments $E(X+r)^p$. Equivalently, it can be used to find the sum

$$S(p) = \sum_{n=0}^{\infty} (n+r)^p \binom{n+r-1}{r-1} x^n,$$

and since $r = m+1$,

$$S(p) = \sum_{n=0}^{\infty} (n+m+1)^p \binom{n+m}{m} x^n.$$

It is clear from the definition of the generating function $\phi(\theta)$ that $S(p) = \phi^{(p)}(0)/(1-x)^r$ for each $p \geq 0$. Dividing equation (9) by $(1-x)^r$ we obtain

$$S(p) = \beta \sum_{j=1}^{p-1} \binom{p}{j} S(j) - \beta \sum_{j=1}^{p-1} \binom{p-1}{j} S(j) + r(1+\beta)S(p-1),$$

which can be written as

$$S(p) = \beta\left(1 + \sum_{j=1}^{p-1} \binom{p}{j} S(j)\right) - \beta\left(1 + \sum_{j=0}^{p-1} \binom{p-1}{j} S(j)\right) + r(1+\beta)S(p-1). \tag{10}$$

This can be used to find $S(p)$ in a recursive manner for any desired p. Conversely, (10) can also be used to determine the moments of the negative binomial distribution. Suppose we want to determine

$$S_1(p) = xS(p) = \sum_{n=0}^{\infty} (n+m+1)^p \binom{n+m}{m} x^{n+1}.$$

It is obvious from (10) that the same recursive equation holds for $S_1(p)$ also. An interesting recursive equation arises if $m = 0$ (i.e., $r = 1$). In this case (10) becomes

$$S_1(p) = \beta\left(1 + \sum_{j=0}^{p-1} \binom{p}{j} S_1(j)\right) - \beta\left(1 + \sum_{j=0}^{p-1} \binom{p-1}{j} S_1(j)\right) + (1+\beta)S_1(p-1).$$

This can be written as

$$S_1(p) = \beta\left(1 + \sum_{j=0}^{p-1} \binom{p}{j} S_1(j)\right) - \beta\left(1 + \sum_{j=0}^{p-2} \binom{p-1}{j} S_1(j)\right) + S_1(p-1), \quad (11)$$

where

$$S_1(p) = \sum_{n=0}^{\infty} (n+1)^p x^{n+1} = \sum_{n=1}^{\infty} n^p x^n.$$

Note that (11) implies

$$S_1(p) = \beta\left(1 + \sum_{j=0}^{p-1} \binom{p}{j} S_1(j)\right) \quad (12)$$

by induction. The recursive formula (12) was derived in [3] by the telescoping method. If $x = 1/2$, then $\beta = 1$, and since $S_1(0) = 1$, it is found that $S_1(1) = 2$, $S_1(2) = 6$, $S_1(3) = 26$, $S_1(4) = 150$, $S_1(5) = 1082$, etc. which were obtained in the preceding section by the combinatorial method.

References

1. E. H. Clarke, "A Formula for the Sum of a Certain Type of Infinite Power Series," *American Mathematical Monthly,* 21 (1914), 292–296.

2. W. Feller, *An Introduction to Probability Theory and its Applications,* Wiley, New York, 1968.

3. R. A. Khan, "Another Elementary Proof of the Convergence-Divergence of the p-Series," *Missouri Journal of Mathematical Sciences,* 16 (2004), 104–107.

4. K. Knopp, *Theory and Applications of Infinite Series,* Blackie & Son Ltd., London, 1928.

5. R. Stalley, “A Generalization of the Geometric Series,” *American Mathematical Monthly,* 56 (1949), 325–327.

Mathematics Subject Classification (2000): 40A25, 65B10; 40A99

Rasul A. Khan
Department of Mathematics
Cleveland State University
Cleveland, OH 44115
email: khan@math.csuohio.edu