ANOTHER LOWER BOUND FOR $\frac{\sin x}{x}$

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The limit

$$\lim_{x \to 0} \frac{\sin x}{x} \tag{$*$}$$

allows us to compute the derivatives of $\sin x$ and $\cos x$. As pointed out by Professor Krantz [1], however, the way it is calculated in most calculus books in print follows a circular argument. In fact, it is customary to use the Pinching Theorem by giving an upper and lower bound for $\frac{\sin x}{x}$. The upper bound

$$\frac{\sin x}{x} \leq 1$$

is trivial as it can be found by comparing the length of a chord with the length of the corresponding arc of a circle. The lower estimate

$$\frac{\sin x}{x} \geq \cos x$$

is usually done assuming the knowledge of the area of a sector, which in turn depends on knowing the area of a circle. To know the area of a circle, however, most calculus books make use of the limit ($*$). To avoid this circular argument, Professor Krantz proposed in [1] the alternative lower bound

$$\frac{\sin x}{x} > \frac{1}{1 + \tan x}.$$

This estimate avoids the use of the area of a circle and uses only the definition of arc length and some elementary geometry. A different estimate was given in [2]. The authors proved the inequalities

$$\sin x \leq x \leq \tan x$$

from which the inequalities

$$\cos x \leq \frac{\sin x}{x} \leq 1$$

follow easily. This was done avoiding the use of the area of a circle and using the definition of length of an arc and some comparisons of linear segments involved in computing the length of an arc of a circle. The purpose of our note is to provide yet another lower estimate for $\frac{\sin x}{x}$. Our estimate is easier to present to first year calculus students. It makes use of the limit

$$\lim_{x \to 0} \frac{1 - \cos x}{x} \tag{$**$}$$

which is used in the calculation of the derivative of $\sin x$ in most calculus books anyway. We also make use of another estimate used in [1], in addition to the trivial comparison of a chord against arc of a circle. In what follows we assume that $0 < x < \frac{\pi}{2}$.

We first include a calculation of $(**)$ to show that our method does not depend on the area of a circle. The following argument is well-known. Since x is larger than m (see Figure 1), we have

$$0 < \frac{1 - \cos x}{x} < \frac{1 - \cos x}{m}.$$

Now

$$\begin{aligned} \frac{1 - \cos x}{m} &= \frac{1 - \cos x}{\sqrt{\sin^2 x + (1 - \cos x)^2}} \\ &= \frac{1 - \cos x}{\sqrt{\sin^2 x + 1 - 2\cos x + \cos^2 x}} \\ &= \frac{1 - \cos x}{\sqrt{2 - 2\cos x}} \\ &= \frac{1}{\sqrt{2}} \sqrt{1 - \cos x}. \end{aligned}$$

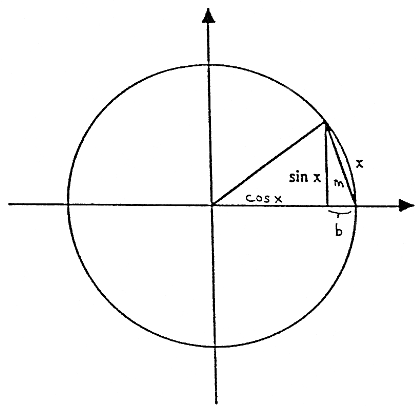

Figure 1.

By the Pinching Theorem,

$$\lim_{x\to 0}\frac{1-\cos x}{x}=0.$$

To compute $(*)$ we will use the estimate

$$\sin x + b > x \tag{$***$}$$

(see Figure 1). This estimate was used in part of the argument in [1] to show

$$\frac{\sin x}{x} > \frac{1}{1+\tan x}.$$

Although $(***)$ is acceptable on heuristic grounds, a careful and simple argument based on the definition of arc length was given in [1] also. To obtain our estimate divide $(***)$ by x and get

$$\frac{\sin x}{x}+\frac{1-\cos x}{x}>1.$$

Take the limit as $x \to 0$ to get

$$\lim_{x\to 0} \frac{\sin x}{x} \geq 1.$$

References

1. S. G. Krantz, "On the Area Inside a Circle," *Missouri Journal of Mathematical Sciences,* 4 (1992), 2–8.

2. R. E. Bayne, J. E. Joseph, and M. H. Kwack, "On the Length of a Circular Arc," *Missouri Journal of Mathematical Sciences,* 11 (1999), 84–86.

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