SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

137. [2002, 210] *Proposed by José Luis Díaz, Universidad Politécnica de Cataluña, Barcelona, Spain.*

Find all non-negative integers a, b, and c such that $a+b+c$ and abc are consecutive integers.

Solution by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri. Assume without loss of generality that $a \leq b \leq c$. If $2 \leq a \leq b \leq c$, then writing $b = a+m$, $c = a+n$ and $abc = a+b+c+1$ implies

$$a^3 + (m+n)a^2 + amn = 3a + m + n + 1.$$

This is a contradiction as the left-hand side is larger than the right-hand side when $a \geq 2$ and m and n are positive integers. We get a similar contradiction when we write $abc = a+b+c-1$. It follows that a has to be either 0 or 1.

Case I. If $a = 0$, then $abc = a+b+c+1$ gives a contradiction. But $abc+1 = a+b+c$ implies $1 = b+c$ which implies $b = 0$ and $c = 1$. So the solution in this case is $a = b = 0$ and $c = 1$.

Case II. If $a = 1$, then either $bc = b+c+2$ or $bc = b+c$.

(i) If $bc = b+c$, then $b = b/c + 1$ and so we must have $b = c$. This implies that $b = 2$ and $c = 2$.

(ii) If $bc = b+c+2$, then $b = (b+2)/c + 1$. This implies that $b+2 = c$. By substitution, $b(b+2) = 2b+4$ or $b^2 = 4$ and so $b = 2$ and $c = 4$.

The solutions are $(0,0,1)$, $(1,2,2)$, and $(1,2,4)$.

Also solved by J. D. Chow, Edinburg, Texas; Joe Howard, Portales, New Mexico; James T. Bruening, Southeast Missouri State University, Cape Girardeau, Missouri; and the proposer.

138. [2002, 210] *Proposed by Joe Howard, Portales, New Mexico.*

Suppose an acute triangle ABC has inradius r and area $\triangle$. Prove

$$\cot A + \cot B + \cot C \geq \frac{\triangle}{3r^2}.$$

Solution I by José Luis Díaz, Universidad Politécnica de Cataluña, Barcelona, Spain. From the Law of Cosines and Sines, we have the cyclic identities

$$\cot A = \frac{\cos A}{\sin A} = \frac{b^2 + c^2 - a^2}{2bc \sin A} = \frac{R}{abc}(b^2 + c^2 - a^2),$$

where R is the circumradius. We must show

$$\cot A + \cot B + \cot C$$

$$= \frac{R}{abc}\big[(b^2 + c^2 - a^2) + (c^2 + a^2 - b^2) + (a^2 + b^2 - c^2)\big] \geq \frac{\triangle}{3r^2}$$

or

$$3(a^2 + b^2 + c^2) \geq \frac{\triangle abc}{Rr^2}. \tag{1}$$

Taking into account that

$$R = \frac{abc}{4\triangle} \quad \text{and} \quad \triangle = sr,$$

where s is the semiperimeter, inequality (1) becomes

$$3(a^2 + b^2 + c^2) \geq (a + b + c)^2. \tag{2}$$

Finally, we need to prove (2). In fact,

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$

$$\leq a^2 + b^2 + c^2 + (a^2 + b^2) + (b^2 + c^2) + (c^2 + a^2) = 3(a^2 + b^2 + c^2).$$

Note that equality holds for an equilateral triangle. This completes the proof.

Solution II by James T. Bruening, Southeast Missouri State University, Cape Girardeau, Missouri. Assume sides a, b, c are opposite angles A, B, C, respectively. Let h_a, h_b, h_c be the altitudes on sides a, b, c, respectively. Since ABC is an acute triangle,

$$\frac{a}{h_a} = \cot B + \cot C, \quad \frac{b}{h_b} = \cot A + \cot C, \quad \text{and} \quad \frac{c}{h_c} = \cot A + \cot B.$$

Thus,

$$\begin{aligned}\cot A + \cot B + \cot C &= \frac{1}{2}\left(\frac{a}{h_a} + \frac{b}{h_b} + \frac{c}{h_c}\right)\\ &= \frac{1}{2}\left(\frac{a}{\frac{2\triangle}{a}} + \frac{b}{\frac{2\triangle}{b}} + \frac{c}{\frac{2\triangle}{c}}\right)\\ &= \frac{a^2+b^2+c^2}{4\triangle}.\end{aligned}$$

The inequality $(a-b)^2 + (a-c)^2 + (b-c)^2 \geq 0$ can be used to derive the inequality

$$a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3}.$$

It can also be shown that $\triangle = sr$, where s is the semiperimeter of the triangle. Applying these above gives

$$\cot A + \cot B + \cot C = \frac{a^2+b^2+c^2}{4\triangle} \geq \frac{(a+b+c)^2}{12\triangle} = \frac{(2s)^2}{12sr} = \frac{s}{3r} = \frac{sr}{3r^2} = \frac{\triangle}{3r^2}.$$

This completes the proof.

Solution III by Mangho Ahuja, Southeast Missouri State University, Cape Girardeau, Missouri. Let E denote the expression

$$E = \cot A + \cot B + \cot C.$$

Then,

$$E = \frac{2bc\cos A}{2bc\sin A} + \frac{2ca\cos B}{2ca\sin B} + \frac{2ab\cos C}{2ab\sin C}.$$

Let s denote the semiperimeter of the triangle ABC. We will use the identities

$$\triangle = \frac{1}{2}ab\sin C, \quad b^2 + c^2 - a^2 = 2bc\cos A, \quad \text{and} \quad \triangle = rs.$$

Then,

$$E = \frac{1}{4\triangle}\left[(b^2 + c^2 - a^2) + (c^2 + a^2 - b^2) + (a^2 + b^2 - c^2)\right] = \frac{1}{4\triangle}(a^2 + b^2 + c^2).$$

Using

$$\sqrt{\frac{a^2 + b^2 + c^2}{3}} \geq \frac{a + b + c}{3},$$

we get

$$E \geq \frac{1}{4\triangle} \cdot \frac{(a + b + c)^2}{3} = \frac{1}{4\triangle} \cdot \frac{4s^2}{3} = \frac{\triangle}{3r^2}.$$

Solution IV by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan. In [1], the following two lemmas are proved.

Lemma 1. In any triangle the following indentity holds.

$$\cot A + \cot B + \cot C = \frac{s^2 - r(4R + r)}{2sr}.$$

Here s is the semiperimeter and R is the circumradius.

Lemma 2. In any triangle the following inequality holds.

$$s^2 \geq 3r^2 + 12Rr.$$

The inequality we wish to prove now reads

$$\frac{s^2 - r(4R + r)}{2sr} \geq \frac{\triangle}{3r^2} = \frac{sr}{3r^2},$$

since $\triangle = sr$. But this inequality is equalivalent to

$$3\big[s^2 - r(4R + r)\big] \geq 2s^2$$

which is equivalent to

$$s^2 \geq 3r^2 + 12Rr.$$

This last inequality holds according to Lemma 2. This completes the proof.

Observation. The above inequality holds in any triangle (not necessarily an acute one).

Reference

1. Constantin C. Florea, *Abordave globala a geometriei triunghiului cu implicatii creative.*

Also solved by J. D. Chow, Edinburg, Texas; Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri; and the proposer.

139. [2002, 210] *Proposed by José Luis Díaz, Universidad Politécnica de Cataluña, Barcelona, Spain.*

Let F_n denote the nth Fibonacci number ($F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$) and let L_n denote the nth Lucas number ($L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$). Show that

$$\sum_{k=1}^{n} L_k^3 = \frac{1}{2}\left(F_{3n+3} + F_{3n+1} + 12(-1)^n F_n + 6(-1)^{n-1} F_{n-1} + 3\right).$$

Solution by James T. Bruening, Southeast Missouri State University, Cape Girardeau, Missouri. Let

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

It is known that the nth Fibonacci number can be represented as

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$$

and the nth Lucas number can be represented as

$$L_n = \alpha^n + \beta^n.$$

These representations can be used to prove the following identities:

$$L_{n-1} = F_n + F_{n-2},$$

$$\sum_{k=1}^{n} (-1)^k L_k = (-1)^n L_{n-1} + 1,$$

$$2 \cdot \sum_{k=1}^{n} L_{3k} = L_{3n+2} - L_2 = L_{3n+2} - 3.$$

Thus,

$$
\begin{aligned}
\sum_{k=1}^{n} L_k^3 &= \sum_{k=1}^{n} (\alpha^k + \beta^k)^3 \\
&= \sum_{k=1}^{n} (\alpha^{3k} + 3\alpha^{2k}\beta^k + 3\alpha^k\beta^{2k} + \beta^{3k}) \\
&= \sum_{k=1}^{n} \big(\alpha^{3k} + \beta^{3k} + 3\alpha^k\beta^k(\alpha^k + \beta^k)\big) \\
&= \sum_{k=1}^{n} \big(\alpha^{3k} + \beta^{3k} + 3(-1)^k(\alpha^k + \beta^k)\big) \\
&= \sum_{k=1}^{n} (L_{3k} + 3(-1)^k L_k) = \sum_{k=1}^{n} L_{3k} + 3\sum_{k=1}^{n} (-1)^k L_k \\
&= \frac{L_{3n+2}}{2} - \frac{3}{2} + \frac{6}{2}\big((-1)^n L_{n-1} + 1\big) \\
&= \frac{1}{2}\big(L_{3n+2} + 6(-1)^n L_{n-1} + 6 - 3\big) \\
&= \frac{1}{2}\big(F_{3n+3} + F_{3n+1} + 6(-1)^n(F_n + F_{n-2}) + 3\big) \\
&= \frac{1}{2}\big(F_{3n+3} + F_{3n+1} + 6(-1)^n(F_n + F_n - F_{n-1}) + 3\big) \\
&= \frac{1}{2}\big(F_{3n+3} + F_{3n+1} + 6(-1)^n(2F_n + F_{n-1}) + 3\big) \\
&= \frac{1}{2}\big(F_{3n+3} + F_{3n+1} + 12(-1)^n F_n + 6(-1)^{n-1} F_{n-1} + 3\big).
\end{aligned}
$$

This completes the proof.

Note. The proposer notes that

$$\sum_{k=1}^{n} F_k^3 = \frac{1}{10}F_{3n+2} + \frac{3}{5}(-1)^{n-1}F_{n-1} + \frac{1}{2}$$

was a problem in [1].

Reference

1. C. Cooper and R. Kennedy, "Problem 3," *Missouri Journal of Mathematical Sciences,* 0 (1988), 29.

Also solved by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri; Joe Howard, Portales, New Mexico; Don Redmond, Southern Illinois University at Carbondale, Carbondale, Illinois; Kenneth B. Davenport, Frackville, Pennsylvania; and the proposer.

140. [2002, 211] *Proposed by José Luis Díaz, Universidad Politécnica de Cataluña, Barcelona, Spain.*

The numbers a, b and c are in geometric progression if and only if

$$(ab + bc + ca)^3 = abc(a + b + c)^3.$$

Prove this.

Solution by Mangho Ahuja, Southeast Missouri State University, Cape Girardeau, Missouri. Let E denote the expression

$$E = abc(a + b + c)^3 - (ab + bc + ca)^3.$$

If we substitute $b^2 = ac$ in E, the expression reduces to zero, hence E has a factor $(b^2 - ac)$. But since E is symmetric in a, b, c; E must have the other two factors $(c^2 - ab)$ and $(a^2 - bc)$ as well. Thus,

$$E = (b^2 - ac)(c^2 - ab)(a^2 - bc)k.$$

On comparing the powers of the polynomial expressions on both sides of this equation, we conclude that k must be a constant. On comparing (on both sides) the coefficients of one of the terms, say a^4bc, we see that k must be 1.

We have thus established that

$$E = (b^2 - ac)(c^2 - ab)(a^2 - bc).$$

It follows that $E = 0$ if and only if one of the three factors is zero, which means the three numbers a, b, c are in geometric progression.

Also solved by James T. Bruening, Southeast Missouri State University, Cape Girardeau, Missouri; J. D. Chow, Edinburg, Texas; Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri; Joe Howard, Portales, New Mexico; and the proposer.