MINIMAL SURFACES: A DERIVATION OF THE MINIMAL SURFACE EQUATION FOR AN ARBITRARY C^2 COORDINATE CHART

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1. Introduction. The study of minimal surfaces is an exciting and active area of mathematical research. Much of the excitement comes from the desire to understand the geometry of soap films, which naturally assume the shapes of minimal surfaces. In this paper, we recall Lagrange's minimal surface equation and derive a more general minimal surface equation. We begin with two important theorems that are well known in the study of minimal surfaces as found in [1].

Theorem 1. (Lagrange, 1760). Let f be a C^2, real-valued function on a planar domain R, and let $\phi: R \to \mathbb{R}^3$ be defined by $\phi(x,y) = (x, y, f(x,y))$. Suppose that the graph of f, i.e., the image $\phi(R)$, is area-minimizing. Then f satisfies the following minimal surface equation.

$$f_{xx}(1+f_y^2) - 2f_x f_y f_{xy} + f_{yy}(1+f_x^2) = 0.$$

A partial converse of this theorem was later proved for specific domains and can be stated as follows.

Theorem 2. (Federer, 1969). Using the notation in Theorem 1, if f satisfies the graph minimal surface equation on a convex domain R, then its graph $\phi(R)$ is area-minimizing.

Each of these theorems is vital to the study of minimal surfaces. However, because of the severe restrictions on the types of coordinate charts, an additional minimal surface equation is necessary for further study. The more general minimal surface equation that would be satisfied by any arbitrary area-minimizing C^2 coordinate chart $\phi: R \to \mathbb{R}^3$ has over 600 terms. (We verified this using Maple©.) Thus, it has never been written down in an easily accessible form. Using the calculus of variations for vector-valued functions, it is this seemingly abominable general minimal surface equation that we derive in this paper and express fairly simply. In addition, we will use this equation to verify that certain surfaces described by general coordinate charts are minimal.

2. Using the calculus of variations, we examine an arbitrary C^2 coordinate chart,

$$\phi(x,y) = (\phi_1(x,y), \phi_2(x,y), \phi_3(x,y)), \quad \phi: R \to \mathbb{R}^3,$$

where R is a bounded, open set in $\mathbb{R}^2$. Let S be the surface in $\mathbb{R}^3$ which is the image $\phi(R)$, and let $A(S)$ be the area of the surface S. Now, suppose we consider a family $\phi(s) = \phi + s\psi$ of such functions, depending on the parameter s, so that $S(s)$ is a family of surfaces. Here $\phi(0) = \phi$, the coordinate chart in question, and $\psi: R \to \mathbb{R}^3$ is a C^2 function such that $\psi|_{\partial R} = 0$. This assumption is equivalent to assuming that the boundary of $S(s)$ is fixed independent of s.

Note that ϕ_x and ϕ_y are vectors in $\mathbb{R}^3$ that also depend on x, y, and s, the variation parameter. Let $F(s) = A(S(s))$, the area function, and assume 0 is a critical point for F for all possible ψ. We express the area as

$$F(s) = \int_R G(\phi_x, \phi_y) dxdy,$$

where $G: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$G(\overrightarrow{u}, \overrightarrow{v}) = \sqrt{\det \begin{pmatrix} \overrightarrow{u} \cdot \overrightarrow{u} & \overrightarrow{u} \cdot \overrightarrow{v} \\ \overrightarrow{v} \cdot \overrightarrow{u} & \overrightarrow{v} \cdot \overrightarrow{v} \end{pmatrix}},$$

for $\overrightarrow{u}, \overrightarrow{v} \in \mathbb{R}^3$. Using the chain rule, we find $F'(0)$, where the prime denotes the partial derivative with respect to s. Since we assume our variations with respect to s are smooth, the derivative $(G(\phi_x, \phi_y))'(s)$ is bounded for every $x, y \in R$ and $0 < s < \epsilon$, $\epsilon > 0$. Therefore, we can differentiate under the integral sign.

$$\begin{aligned} F'(0) &= \int_R (G(\phi_x, \phi_y))'(0) dxdy \\ &= \int_R \nabla G(\phi_x, \phi_y) \cdot (\phi_x, \phi_y)'(0) dxdy \\ &= \int_R \Big(\nabla_{\overrightarrow{u}} G(\phi_x, \phi_y) \cdot (\phi_x)'(0) + \nabla_{\overrightarrow{v}} G(\phi_x, \phi_y) \cdot (\phi_y)'(0) \Big) dxdy. \end{aligned} \tag{1}$$

Here, $\nabla_{\overrightarrow{u}} G$ denotes the gradient of the function $G(\cdot, \overrightarrow{v}) \colon \mathbb{R}^3 \to \mathbb{R}$ for constant $\overrightarrow{v}$ and $\nabla_{\overrightarrow{v}} G$ denotes the gradient of the function $G(\overrightarrow{u}, \cdot)$ for constant $\overrightarrow{u}$. Thus, $\nabla G = \nabla_{\overrightarrow{u}} G + \nabla_{\overrightarrow{v}} G$.

Recall that, $\phi_x(s) = (\phi + s\psi)_x$ and $\phi_y(s) = (\phi + s\psi)_y$. Therefore, $(\phi_x)'(0) = \psi_x$ and $(\phi_y)'(0) = \psi_y$. Substituting these into equation (1), we see

$$F'(0) = \int_R (\nabla_{\overrightarrow{u}} G(\phi_x, \phi_y) \cdot \psi_x + \nabla_{\overrightarrow{v}} G(\phi_x, \phi_y) \cdot \psi_y) dxdy.$$

We integrate by parts in the x variable in the first term and in the y variable in the second term and factor out the function ψ.

$$F'(0) = \int_R \left(-(\nabla_{\overrightarrow{u}} G(\phi_x, \phi_y))_x \cdot \psi - (\nabla_{\overrightarrow{v}} G(\phi_x, \phi_y))_y \cdot \psi \right) dxdy. \tag{2}$$

Since $\psi = 0$ on ∂R, the boundary terms in the integration by parts equal zero. Factoring equation (2), we have

$$F'(0) = -\int_R \left[(\nabla_{\overrightarrow{u}} G(\phi_x, \phi_y))_x + (\nabla_{\overrightarrow{v}} G(\phi_x, \phi_y))_y \right] \cdot \psi \; dxdy.$$

Recall that since $F(s)$ is the area function and we assumed 0 was a critical point of F for all possible ψ, $F'(0)$ must equal 0 for all possible ψ. In particular, if we choose

$$\psi = (\nabla_{\overrightarrow{u}} G(\phi_x, \phi_y))_x + (\nabla_{\overrightarrow{v}} G(\phi_x, \phi_y))_y,$$

then

$$\left\| \left[(\nabla_{\overrightarrow{u}} G(\phi_x, \phi_y))_x + (\nabla_{\overrightarrow{v}} G(\phi_x, \phi_y))_y \right] \right\|^2 = 0.$$

This implies that

$$(\nabla_{\overrightarrow{u}} G(\phi_x, \phi_y))_x + (\nabla_{\overrightarrow{v}} G(\phi_x, \phi_y))_y = 0. \tag{3}$$

Now, we substitute for G. Let $\overrightarrow{u}, \overrightarrow{v} \in \mathbb{R}^3$ be given, and let θ be the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$. Note that $0 \leq \theta \leq \pi$.

$$(G(\overrightarrow{u}, \overrightarrow{v}))^2 = \det \begin{pmatrix} \overrightarrow{u} \cdot \overrightarrow{u} & \overrightarrow{u} \cdot \overrightarrow{v} \\ \overrightarrow{v} \cdot \overrightarrow{u} & \overrightarrow{v} \cdot \overrightarrow{v} \end{pmatrix}$$

$$\begin{aligned} &= (\overrightarrow{u} \cdot \overrightarrow{u})(\overrightarrow{v} \cdot \overrightarrow{v}) - (\overrightarrow{u} \cdot \overrightarrow{v})(\overrightarrow{v} \cdot \overrightarrow{u}) \\ &= \|\overrightarrow{u}\|^2 \|\overrightarrow{v}\|^2 - \|\overrightarrow{u}\|^2 \|\overrightarrow{v}\|^2 \cos^2 \theta \\ &= \|\overrightarrow{u}\|^2 \|\overrightarrow{v}\|^2 (\sin^2 \theta) \\ &= \|\overrightarrow{u} \times \overrightarrow{v}\|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} G(\overrightarrow{u}, \overrightarrow{v}) &= \|\overrightarrow{u} \times \overrightarrow{v}\| \\ &= (\langle \overrightarrow{u} \times \overrightarrow{v}, \overrightarrow{u} \times \overrightarrow{v} \rangle)^{1/2}. \end{aligned}$$

We calculate the first term of equation (3). For every $\overrightarrow{u}, \overrightarrow{v} \in \mathbb{R}^3$,

$$\nabla_{\overrightarrow{u}} G(\overrightarrow{u}, \overrightarrow{v}) = \nabla_{\overrightarrow{u}} (\langle \overrightarrow{u} \times \overrightarrow{v}, \overrightarrow{u} \times \overrightarrow{v} \rangle)^{1/2}$$

$$= \Bigg(\frac{\partial}{\partial u_1} (\langle \overrightarrow{u} \times \overrightarrow{v}, \overrightarrow{u} \times \overrightarrow{v} \rangle)^{1/2},$$

$$\frac{\partial}{\partial u_2} (\langle \overrightarrow{u} \times \overrightarrow{v}, \overrightarrow{u} \times \overrightarrow{v} \rangle)^{1/2}, \frac{\partial}{\partial u_3} (\langle \overrightarrow{u} \times \overrightarrow{v}, \overrightarrow{u} \times \overrightarrow{v} \rangle)^{1/2} \Bigg)$$

$$= \Bigg((\langle \overrightarrow{u} \times \overrightarrow{v}, \overrightarrow{u} \times \overrightarrow{v} \rangle)^{-1/2} (2(u_3 v_1 - u_1 v_3)(-v_3) + 2(u_1 v_2 - u_2 v_1)(v_2)),$$

$$(\langle \overrightarrow{u} \times \overrightarrow{v}, \overrightarrow{u} \times \overrightarrow{v} \rangle)^{-1/2} (2(u_2 v_3 - u_3 v_2)(v_3) + 2(u_1 v_2 - u_2 v_1)(-v_1)),$$

$$(\langle \overrightarrow{u} \times \overrightarrow{v}, \overrightarrow{u} \times \overrightarrow{v} \rangle)^{-1/2} (2(u_2 v_3 - u_3 v_2)(-v_2) + 2(u_3 v_1 - u_1 v_3)(v_1)) \Bigg)$$

$$= (\langle \overrightarrow{u} \times \overrightarrow{v}, \overrightarrow{u} \times \overrightarrow{v} \rangle)^{-1/2} \Big([-(\overrightarrow{u} \times \overrightarrow{v})_2 v_3 + (\overrightarrow{u} \times \overrightarrow{v})_3 v_2],$$

$$[(\overrightarrow{u} \times \overrightarrow{v})_1 v_3 - (\overrightarrow{u} \times \overrightarrow{v})_3 v_1], [-(\overrightarrow{u} \times \overrightarrow{v})_1 v_2 + (\overrightarrow{u} \times \overrightarrow{v})_2 v_1] \Big)$$

$$= (\langle \overrightarrow{u} \times \overrightarrow{v}, \overrightarrow{u} \times \overrightarrow{v} \rangle)^{-1/2} (\overrightarrow{v} \times (\overrightarrow{u} \times \overrightarrow{v})).$$

Thus,

$$\nabla_{\overrightarrow{u}} G(\phi_x, \phi_y) = (\langle \phi_x \times \phi_y, \phi_x \times \phi_y \rangle)^{-1/2} (\phi_y \times (\phi_x \times \phi_y)). \tag{4}$$

By symmetry, we find the second term of equation (3).

$$\begin{aligned} \nabla_{\overrightarrow{v}} G(\overrightarrow{u}, \overrightarrow{v}) &= \nabla_{\overrightarrow{v}} (\langle \overrightarrow{u} \times \overrightarrow{v}, \overrightarrow{u} \times \overrightarrow{v} \rangle)^{1/2} \\ &= (\langle \overrightarrow{u} \times \overrightarrow{v}, \overrightarrow{u} \times \overrightarrow{v} \rangle)^{-1/2} (\overrightarrow{u} \times (\overrightarrow{v} \times \overrightarrow{u})). \end{aligned}$$

Thus,

$$\nabla_{\overrightarrow{v}} G(\phi_x, \phi_y) = (\langle \phi_x \times \phi_y, \phi_x \times \phi_y \rangle)^{-1/2} (\phi_x \times (\phi_y \times \phi_x)). \tag{5}$$

Substituting equations (4) and (5) into equation (3), we have the following theorem.

Theorem 3. Let R be a bounded, open set in $\mathbb{R}^2$, and let $\phi\colon R \to \mathbb{R}^3$ be a one-to-one, C^2, vector-valued function such that $\phi(R)$ is area-minimizing. Then, ϕ satisfies the general minimal surface equation.

$$\left(\frac{\phi_y \times (\phi_x \times \phi_y)}{\|\phi_x \times \phi_y\|} \right)_x + \left(\frac{\phi_x \times (\phi_y \times \phi_x)}{\|\phi_x \times \phi_y\|} \right)_y = 0.$$

3. In this section, we look at some familiar results that appear as consequences of Theorem 3.

Example 1. Lagrange's Minimal Surface Equation. We begin with the chart $\phi(x, y) = (x, y, f(x, y))$, where f is a C^2, real-valued function on a planar domain

R. Then $\phi_x = (1, 0, f_x)$ and $\phi_y = (0, 1, f_y)$. Using Maple© to calculate the cross products and derivatives in Theorem 3, we obtain

$$\Bigg(\frac{-f_x(-2f_xf_yf_{xy} + f_{xx}(1+f_y^2) + f_{yy}(1+f_x^2))}{(1+f_x^2+f_y^2)^{3/2}},$$

$$\frac{-f_y(-2f_xf_yf_{xy} + f_{xx}(1+f_y^2) + f_{yy}(1+f_x^2))}{(1+f_x^2+f_y^2)^{3/2}},$$

$$\frac{-2f_xf_yf_{xy} + f_{xx}(1+f_y^2) + f_{yy}(1+f_x^2)}{(1+f_x^2+f_y^2)^{3/2}}\Bigg) = 0.$$

Simplifying,

$$(f_{xx}(1+f_y^2) - 2f_xf_yf_{xy} + f_{yy}(1+f_x^2))\frac{(-f_x, -f_y, 1)}{(1+f_x^2+f_y^2)^{3/2}} = 0. \tag{6}$$

Since the vector $(-f_x, -f_y, 1)$ in equation (6) is non-zero, the scalar to the left must equal 0, which gives us Lagrange's graph minimal surface equation.

The vector $(-f_x, -f_y, 1)$ we obtain is interesting itself. Let $z = f(x, y)$ and define a function h as follows.

$$h(x, y, z) = z - f(x, y).$$

Level sets for the function h are of the form $h(x, y, z) = c$ for some $c \in \mathbb{R}$ and the level set for $c = 0$ is our minimal surface. The gradient $\nabla h(x_0, y_0, z_0)$ is perpendicular to the level set $h(x, y, z) = 0$ provided $\nabla h(x_0, y_0, z_0) \neq 0$. In our case, $\nabla h = (-f_x, -f_y, 1)$, which is the vector we see in equation (6). Now, the vector $(-f_x, -f_y, 1)$ is normal to our surface at $(x, y, f(x, y))$. Thus, if we let $\overrightarrow{N} = (-f_x, -f_y, 1)$, we can rewrite equation (6) as

$$(f_{xx}(1+f_y^2) - 2f_xf_yf_{xy} + f_{yy}(1+f_x^2))\frac{\overrightarrow{N}}{\|\overrightarrow{N}\|^{3/2}} = 0.$$

Example 2. Conformal Coordinates. A conformal map $\phi\colon R \to \mathbb{R}^3$ is a map such that $\phi_x \cdot \phi_x = \phi_y \cdot \phi_y$ and $\phi_x \cdot \phi_y = 0$. If ϕ is conformal and $\phi(R)$ is minimal, then ϕ satisfies Laplace's equation $\phi_{xx} + \phi_{yy} = 0$ (see, for example, [4]). This means that every coordinate function satisfies Laplace's equation. Here, we show that if we assume ϕ is a conformal map and substitute ϕ into Theorem 3, the result will be Laplace's equation. For notational simplification, let $\overrightarrow{u} = \phi_x$ and $\overrightarrow{v} = \phi_y$. Let us begin with the first term of Theorem 3. Let $\hat{w}$ be the unit vector in the $\overrightarrow{u} \times \overrightarrow{v}$ direction. Then $\overrightarrow{u} \times \overrightarrow{v} = \|\overrightarrow{u}\|\,\|\overrightarrow{v}\|\hat{w}$, since $\overrightarrow{u}$ and $\overrightarrow{v}$ are perpendicular. Now, $\overrightarrow{v} \times (\|\overrightarrow{u}\|\,\|\overrightarrow{v}\|\hat{w})$ is in the direction of $\overrightarrow{u}$, and its magnitude is $\|\overrightarrow{v}\|\,\|\overrightarrow{u}\|\,\|\overrightarrow{v}\|$, again because the vectors $\overrightarrow{v}$ and $\hat{w}$ are perpendicular. Therefore, the entire vector can be expressed as

$$\overrightarrow{v} \times (\overrightarrow{u} \times \overrightarrow{v}) = \|\overrightarrow{v}\|^2 \|\overrightarrow{u}\| \frac{\overrightarrow{u}}{\|\overrightarrow{u}\|},$$

where $\overrightarrow{u}/\|\overrightarrow{u}\|$ is the unit vector in the $\overrightarrow{u}$ direction, so $\overrightarrow{v} \times (\overrightarrow{u} \times \overrightarrow{v}) = \|\overrightarrow{v}\|^2 \overrightarrow{u}$. Substituting into the fraction in the first term of Theorem 3,

$$\begin{aligned}\left(\frac{\overrightarrow{v} \times (\overrightarrow{u} \times \overrightarrow{v})}{\|\overrightarrow{u} \times \overrightarrow{v}\|}\right)_x &= \left(\frac{\|\overrightarrow{v}\|^2 \overrightarrow{u}}{\|\overrightarrow{u} \times \overrightarrow{v}\|}\right)_x \\ &= \left(\frac{\|\overrightarrow{v}\|^2 \overrightarrow{u}}{\|\overrightarrow{u}\|\,\|\overrightarrow{v}\|}\right)_x \\ &= \left(\frac{\|\overrightarrow{v}\|}{\|\overrightarrow{u}\|}\overrightarrow{u}\right)_x.\end{aligned}$$

By the conformal hypothesis, $\|\overrightarrow{u}\| = \|\overrightarrow{v}\|$ and thus,

$$\left(\frac{\overrightarrow{v} \times (\overrightarrow{u} \times \overrightarrow{v})}{\|\overrightarrow{u} \times \overrightarrow{v}\|}\right)_x = (\overrightarrow{u})_x.$$

By similar calculations, we see that the second term of Theorem 3 becomes

$$\left(\frac{\overrightarrow{u} \times (\overrightarrow{v} \times \overrightarrow{u})}{\|\overrightarrow{u} \times \overrightarrow{v}\|}\right)_y = (\overrightarrow{v})_y.$$

Thus, the entire expression in Theorem 3 becomes $(\overrightarrow{u})_x + (\overrightarrow{v})_y = 0$, which by our substitutions of $\overrightarrow{u} = \phi_x$ and $\overrightarrow{v} = \phi_y$ becomes Laplace's equation, $\phi_{xx} + \phi_{yy} = 0$.

Example 3. The Catenoid. Consider a parameterization of the catenoid as in [3].

$$\phi(u,v) = (r(u)\cos v, r(u)\sin v, u), \qquad r(u) = \cosh u.$$

Note that this gives a catenoid whose axis of revolution is the z axis. Then,

$$\begin{aligned}
\phi_u &= (\sinh(u)\cos v, \sinh(u)\sin v, 1),\\
\phi_v &= (-\cosh(u)\sin v, \cosh(u)\cos v, 0),\\
\phi_u \times \phi_v &= (-\cosh(u)\cos v, -\cosh(u)\sin v, \sinh(u)\cosh(u)), \text{ and}\\
\|\phi_u \times \phi_v\| &= \cosh^2(u).
\end{aligned}$$

It is easily shown that this ϕ satisfies Theorem 3.

Example 4. Generalized Catenoid. Consider the twisted catenoid, as given by the parameterization in [2].

$$\begin{aligned}
\phi_1(u,v) &= u\cos v\\
\phi_2(u,v) &= m(u) + u\sin v\\
\phi_3(u,v) &= \int_1^u \frac{dw}{\triangle(w)},
\end{aligned}$$

where

$$m(u) = \int_1^u \frac{w^2 dw}{\triangle(w)}$$

and

$$\triangle(w) = \sqrt{w^4 - 1}.$$

Thus, we have the following equations.

$$\phi_u = \left(\cos v, \frac{u^2 + \sin v\sqrt{u^4-1}}{\sqrt{u^4-1}}, \frac{1}{\sqrt{u^4-1}}\right)$$

$$\phi_v = (-u\sin v, u\cos v, 0)$$

$$\phi_u \times \phi_v = \frac{1}{\sqrt{u^4-1}}\left(-u\cos v, -u\sin v, \frac{u(u^2\sin v + \sqrt{u^4-1})}{\sqrt{u^4-1}}\right)$$

$$\|\phi_u \times \phi_v\| = \frac{u}{u^4-1}\left(2u^4 - 2 + u^4\sin^2 v + 2u^2(\sin v)\sqrt{u^4-1}\right)^{1/2}.$$

After lengthy computations, we see that this ϕ satisfies Theorem 3; therefore, the image of ϕ is a minimal surface. Although the computation is lengthy, the use of Theorem 3 is noteworthy because the verification by Riemann was not done by explicit computation. Note that it is not possible to use Lagrange's equation to verify that the parameterization given by ϕ is a minimal surface.

References

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