GRAPHIC REPRESENTATIONS FOR ASSOCIATIVE ALGEBRAS

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1. Introduction. In mathematics, many problems become easier to solve when it is possible to have some kind of graphic representation either of the problem itself, or of the objects which are involved in it. By that way, the searcher (or the student) can get a geometric intuition of what he/she is studying. Associative algebras arise in many fields of mathematics and their study is a very live field of research. Therefore we present here a construction of some graphs associated to an associative algebra with unit which provide this intuition.

2. Peirce Decomposition of an Algebra. Let A be an associative algebra with unit 1 over a sufficiently large algebraically closed field K. In this note, we will call A an algebra. In fact, A is a vector space over K. Since we have taken K sufficiently large, a theorem of Wedderburn implies that, as a vector space, A is the direct sum of a (non-unique) subalgebra S and an ideal J, i.e. $A = S \oplus J$. The subalgebra S is a direct sum of matrix blocks $S \oplus_i M_{n_i}(K)$, and J is an ideal, i.e. $aJ \subseteq J$ and $Ja \subseteq J$ for any $a \in A$. The subalgebra S is called a separable subalgebra of A and J is called the radical of A.

If e is an idempotent element in A, i.e. $e^2 = e$, then every element in A can be written in the form $a = eae + ea(1-e) + (1-e)ae + (1-e)a(1-e)$. Moreover, if we write $xAy = \{xay\ ;\ a \in A\}$, it is clear that we have the following decomposition of the algebra A:

$$A = eAe \oplus eA(1-e) \oplus (1-e)Ae \oplus (1-e)A(1-e).$$

This decomposition is called the two-sided Peirce decomposition of A with respect to e. The subspaces eAe, $eA(1-e)$, $(1-e)Ae$ and $(1-e)A(1-e)$ are called the Peirce components of A with respect to e.

Proposition 2.1. Let A be an arbitrary algebra. If J is the radical of A, then $eJe = eAe \cap J$, $eJ(1-e) = eA(1-e) \cap J$, $(1-e)Je = (1-e)Ae \cap J$, $(1-e)J(1-e) = (1-e)A(1-e) \cap J$.

For a proof. see [5].

Now let A be an algebra. Two idempotents e and f are orthogonal if $ef = fe = 0$; an idempotent e is primitive if it cannot be written as the sum of two non-zero idempotents. A family $e_1, e_2, \ldots, e_r$ of idempotents is a family of orthogonal idempotents if they are pairwise orthogonal, i.e. $e_i e_j = 0$ for every $i \neq j$. The family is complete if $\sum e_i = 1$.

The Peirce decomposition can be generalized in the following way: let e_1, e_2, $\ldots$, e_r be a finite set of idempotents such that $\sum e_i = 1$; then $A = \oplus_{i,j} e_i A e_j$. The terms of this sum are also called Peirce components. More details can be found in [5].

Figure 1. Two simple basis-graphs

3. Basis-Graphs. We consider now the following directed graph associated to a finite dimensional algebra A:

1. Choose a complete set of primitive orthogonal idempotents e_1, e_2, $\ldots$, e_r. The number of vertices of the graph is the number of idempotents in the set; label each vertex by an idempotent.
2. For $i \neq j$, the number of arrows from e_i to e_j is equal to $\dim_K(e_i A e_j)$.
3. The number of loops from e_i to itself is equal to $\dim_K(e_i A e_i) - 1$.

This graph is called the basis-graph of the algebra A.

Example 3.1. In Figure 1 we show basis-graphs for (i) upper-triangular matrices of order 3 and (ii) $K[x]/(x^3)$.

For (i) we chose as idempotents the matrices E_{11}, E_{22}, E_{33}, and the arrows represent the matrices E_{12}, E_{23}, E_{13}. Recall that E_{ij} is the matrix with an (i,j)-entry equal to 1 and 0 elsewhere. For (ii) the idempotent is 1, the loops represent x and x^2 respectively.

Remark 3.2. In representation theory of associative algebras one would use another graph, named quiver, which contains arrows only for J/J^2, where J is the radical of the algebra. For instance, all the algebras $K[x]/(x^n)$ have the same quiver, which contains one vertex and one loop.

In fact, we can see that the basis-graph in (ii) can represent another algebra, namely $K[x,y]/(x^2,y^2,xy)$. We need to refine the definition in order to distinguish various algebras with the same basis-graphs.

Let A be an algebra with radical J. The weighted basis-graph associated with A is the basis-graph of A with the arrows and loops weighted as follows:

1. The number of arrows from e_i to e_j with weight k is equal to the number $n_{ij}^k = \dim_K e_i(J^k/J^{k+1})e_j$. A k-weighted arrow is an arrow with k barbs.
2. Matrix units are weighted by infinity and marked by a solid triangular barb.
3. A trivial loop, i.e. a loop with only trivial products with other basis elements, will be marked by a "circle" on it.
4. Products of arrows are assumed to be zero, unless identified with a particular arrow or loop.

Example 3.3. In Figure 2 weighted basis-graphs are displayed for (i) upper-triangular matrices of order 3, (ii) $K[x,y]/(x,y)^2$ and (iii) $K[x]/(x^3)$.

Example 3.4. Figure 3 displays weighted basis-graph for 2×2-matrices and 3×3-matrices.

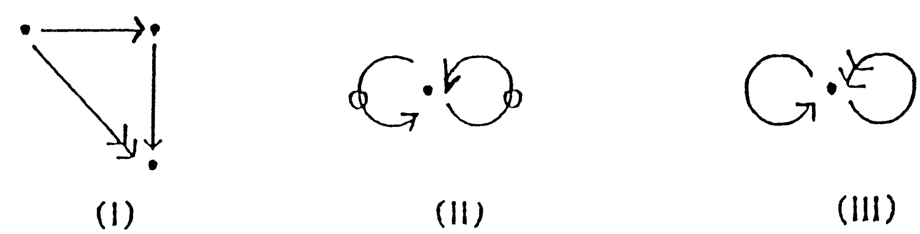

Figure 2.

Figure 3. Matrices.

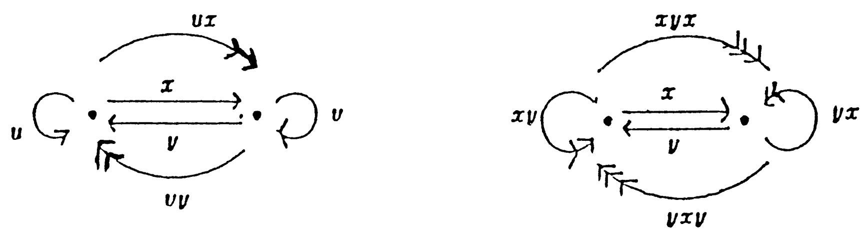

Figure 4. Two different weighings.

Example 3.5. In Figure 4 we give an example of a basis-graph with two different weighings. The weighted basis-graph on the left can be the basis-graph of an algebra with additional relations among basis elements (e.g. $ux = xv$, $vy = yu$). It is not difficult to define an algebra isomorphism between this new algebra and the algebra corresponding to the displayed basis-graph.

With a weighted basis-graph we have a "complete picture" of a basis of the algebra filtered by the powers of the radical. Despite the fact that the construction is in no way canonical, two algebras which are isomorphic have the same weighted basis-graph. Therefore, establishing a complete list of algebras in a given dimension n (up to isomorphism) is now partly combinatorial work. As an example, we give now the complete lists of associative algebras with units in dimension 2, 3, and 4. We denote by $K[x_1, \dots, x_n]$ the polynomial ring with n commuting variables, and by $K\langle x_1, \dots, x_n\rangle$ the polynomial ring with n non-commuting variables. The algebras are described in the text from the left to the right.

1. Two-dimensional algebras. Any algebra is isomorphic either to K^2 or to $K[x]/(x^2)$ (see Figure 5).
2. Three-dimensional algebras. Any algebra is isomorphic either to K^3, to $K[x,y]/(x,y)^2$ or to $K[x]/(x^3)$ or to the upper triangular matrices (see Figure 6).

Figure 5. 2-dimensional algebras.

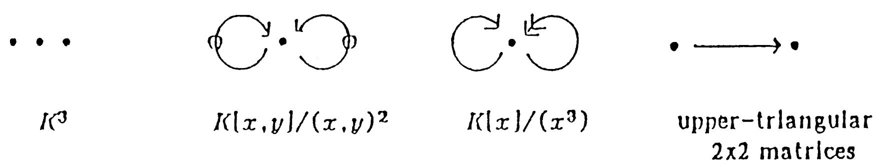

Figure 6. 3-dimensional algebras.

Figure 7. 4-dimensional algebras with 3 idempotents.

3. Four-dimensional algebras. We will list the algebras according to decreasing number of idempotents:

   (a) With four idempotents, there is only one algebra, namely K^4.

   (b) With three idempotents, there are two algebras, namely $K \times T_2(K)$ and $K^2 \times K[x]/(x^2)$, where $T_2(K)$ denotes the algebra of upper triangular matrices of order 2 over K. For these two cases, see Figure 7.

   (c) With two idempotents, we have the following algebras, whose respective weighted basis-graphs are given in Figure 8:

- $K[x]/(x^2) \times K[x]/(x^2)$.
- $K \times K[x,y]/(x,y)^2$.
- $K \times K[x]/(x^3)$.
- The algebra of 4×4 matrices of the form

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & d \\ c & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$$

  where $a, b, c, d \in K$.

- The algebra $M_2(K)$ of all square matrices of order 2.
- The algebra of 3×3 matrices of the form

$$\begin{pmatrix} a & c & d \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$$

  where $a, b, c \in K$, and its dual.

- The Kronecker algebra, whose elements are the 3×3 matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & d \end{pmatrix}$$

where $a, b, c, d \in K$.

(d) The algebras with one idempotent, i.e. the local algebras are $K[x,y,z]/(x,y,z)^2$, $K[x,y]/(x^2, xy, y^3)$, $K[x]/(x^4)$, $K\langle x,y\rangle/(x^2, y^2, yx - sxy)$, $s \neq 0, 1$ and $K[x,y]/(x^3, x^2 + yx, xy + yx, y^2)$, as displayed in Figure 9.

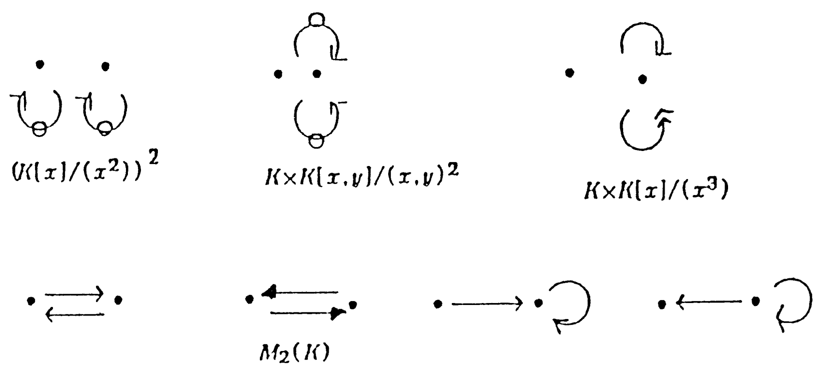

Figure 8. 4-dimensional algebras with 2 idempotents.

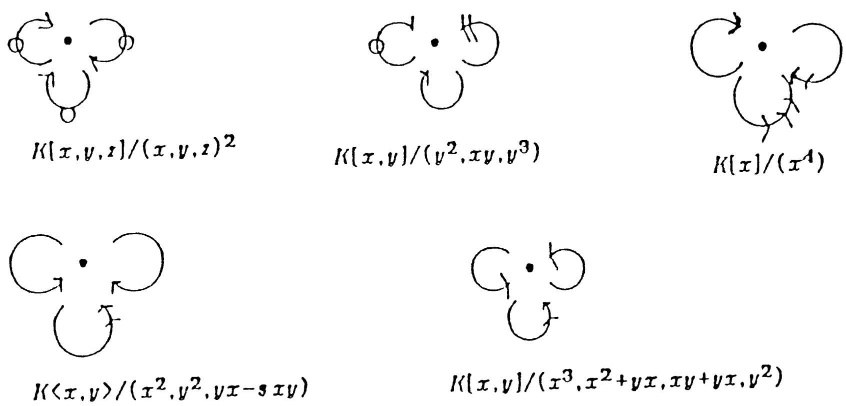

Figure 9. 4-dimensional algebras with 1 idempotent.

A detailed list can be found in [3] (we took some notations from it); there the list is longer as Gabriel distinguished between some algebras or parameterized families with basis-graph containing one idempotent and 3 loops, for the sake of deformation. For local 4-dimensional algebras, we got a one-parameter family of non- isomorphic algebras with the same weighted basis-graph, namely $K\langle x,y\rangle/(x^2,y^2,yx-sxy)$.

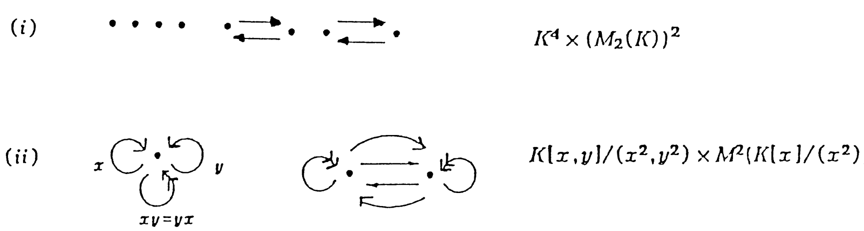

Figure 10. Group algebras.

Remark 3.6. It is easy to understand that the basis-graph of a commutative algebra contains only vertices and/or loops, but no non-looped arrow. In a commutative algebra, $e_ixe_j = e_ie_jx$. If $i \neq j$, then $e_ie_j = 0$, so for the Peirce components we get $e_iAe_j = e_ie_jA = 0$. In the examples of 3-dimensional algebras, the only non-commutative algebra is the algebra of upper triangular 2×2 matrices (for an algebraic proof of the fact that up to isomorphism this is the only non-commutative algebra, see [4]).

A considerable work has been done using these graphs in deformation theory of associative algebras with unit, as their behaviour under deformation is well-known, and well suited (see [6]). If an algebra B is a deformation of an algebra A, then the basis-graph of A is either the same as the basis-graph of B or it is obtained from the basis-graph of B by coalescing vertices, adding a new loop for each vanishing vertex. For example, the local algebra with one loop $K[x]/(x^2)$ deforms to K^2, whose basis-graph contains only two vertices. These techniques made possible a classification of low dimension associative algebras with unit (e.g. see [1,8]). For loopless basis-graphs, the classification is known up to dimension 9, for the other cases, there exists upper bounds for the number of irreducible components of the scheme Alg_n that classify them.

The basis-graph of an algebra is also used in studying the Donald-Flanigan problem of deforming modular group algebras (i.e. group algebra of a finite group over a field whose characteristic divides the order of the group) into semi-simple algebras (e.g. see [7]). The following and last example we give here displays the weighted basis-graphs of the group algebra KG, where $G = D_6$ is the dihedral group $D_6 = \{a, b \mid a^6 = b^2 = e,\ ab = ba^{-1}\}$ and the characteristic of the field K is (i) 0 and (ii) 2 (see Figure 10).

(i) In characteristic 0, KD_6 is composed of 4 blocks isomorphic to K and two blocks of 2×2-matrices.

(ii) In characteristic 2, KD_6 contains one local block and one matrix block over $K[x]/(x^2)$.

In characteristic either 0 or non-dividing the order of the group, finding the basis-graph of the group algebra is not difficult for small groups. When the characteristic of the field divides the order of the group, the problem is much more complicated and lays beyond the scope of the present paper.

References

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