ON RESOLVING THE LITTLEWOOD-ROSS PARADOX

John Byl

Abstract. In this paper the Littlewood-Ross paradox is discussed. Some questions are raised regarding the most common resolution of this paradox and an alternative resolution is proposed.

1. Introduction. The eminent mathematician J. E. Littlewood described the following paradox of the infinite [5]. Balls numbered $1, 2, \ldots$ are put into an urn as follows. At 1 minute to noon the balls numbered 1 to 10 are put in, and the number 1 is taken out. At $1/2$ minute to noon numbers 11 to 20 are put in and the number 2 is taken out. At $1/3$ minute to noon 21 to 30 are put in and 3 is taken out. And so on. How many balls are in the urn at noon?

At first sight one might think that the number N of balls in the urn at noon should be infinite, since we have an infinite number of operations, each adding a net total of 9 balls to the urn. It seems that we should have

$$N = \lim_{n \to \infty} 9n = \infty. \tag{1}$$

On the other hand, since ball n is removed at the nth operation, and since n takes on all values from 1 to infinity, it seems that the urn should be empty at noon! An intriguing paradox.

Which answer is the correct one? Littlewood placed more weight on the second line of argumentation: any selected number, say 106, is absent since it was taken out at the 106th operation. Hence, Littlewood concluded that at noon the urn is empty. The same paradox has been described more recently by Ross [6] who, using similar reasoning, reached the same verdict.

During the past few years this paradox has been discussed by various authors. Allis & Koetsier [1,2] and Earman & Norton [3] concurred that the urn is empty at noon; Holgate [4] added some mathematical comments; and van Bendegem [7] reasoned that, since two contradictory conclusions are reached (i.e., the urn at noon is both empty and full), the super-task (i.e., an infinite sequence of actions) described by Littlewood is impossible.

As noted by Allis & Koetsier [2], paradoxes of the infinite are deceptive research topics. Often we are led astray by intuitions and reasoning patterns from which the flawed ingredients can be extracted only through thorough investigation.

In this paper we examine some difficulties associated with the proposal of an empty urn at noon, and argue for the alternative interpretation of an infinite number of balls in the urn at noon.

2. Continuity Condition. One difficulty with the paradox, as has been pointed out by Allis & Koetsier [1] and Earman & Norton [3], is that the problem is logically underdescribed. The problem as posed merely describes what happens at a number of particular instances *before* noon. No constraints have been placed on what happens *at or after* noon. Thus, logically anything could happen.

What is needed is the specification of some further, physically plausible, continuity conditions. Two such apparently quite natural conditions can be applied to fix the number of balls in the urn at noon.

(1) The principle adopted by Allis & Koetsier [1] is that the position functions of the balls are continuous functions of time. If at some moment before noon a ball comes to rest in a particular position, which it does not leave before noon, then it is considered to be still at that position at noon.

(2) Another possibility is that the number $N(t)$ of balls in the urn is continuous at any time t at which no ball is added or subtracted from the vase. Thus, at such a time t, $N(t) = \lim_{t' \to t} N(t')$.

For continuity condition (2), since no balls are added or subtracted at noon, the urn at that time clearly contains an infinite number of balls. Condition (1), on the other hand, is asserted by Allis & Koetsier to lead to an empty noon-time urn.

Which continuity condition is more plausible? Earman & Norton prefer the continuity of the position functions on the grounds that this is favored by the numbering of the balls, whereby they retain their individual identity through time. Also, they argue that it refers to the simplest spacetime picture for the kinematics of the balls.

One wonders, however, whether these two conditions are necessarily incompatible. According to Earman & Norton, the assumption of continuous position functions leads to the conclusion that the number function $N(t)$ increases without limit with each stage as noon is approached, whereupon it falls discontinuously to zero.

Consider, however, what such a discontinuity in the number function entails. The instant before noon the urn contains an infinity of distinct balls, each with its unique label and each with its individual position function. At noon these balls are considered to be instantaneously ejected from the urn. Viewed kinematically, this involves moving a finite distance in zero time, causing discontinuities to appear in the corresponding position functions. It seems, therefore, that the continuity of all position functions necessarily implies also the continuity of the number function.

Indeed, if we follow Allis & Koetsier in assuming that any ball coming to rest at a position which it does not leave before noon is still at that position at noon, then all balls in the urn just before noon should still be there at noon. (Here I am assuming that a ball placed in the urn remains in the same position until it exits).

In this case the continuity of position functions seems to lead to the same result as does the continuity of the number function: a full urn at noon.

In short, it seems quite natural to conclude that the two continuity conditions are in fact consistent. Moreover, both continuity conditions seem to suggest that the urn is full at noon.

3. Problems with an Empty Urn. There is a further difficulty with the conclusion that at noon the urn is empty. What causes the mysterious evaporation of the contents of the urn at noon? That the urn should be empty at noon seems amazing, since the number of balls in the urn is steadily increasing as noon is approached. The super-task consists of an infinite number of similar actions, each action consisting of adding 10 balls and subtracting one, for a net addition of 9 more balls to the urn. How, then, can the urn be suddenly empty as the hour strikes? How can an infinity of net additions be completely canceled if not a single action involves a net subtraction?

Earman & Norton assert that this is simply the artifact of the subtraction of one infinite set from another. Yet there must be more to it than that. First, the subtraction of one infinite set from another does not necessarily lead to a null result; it depends on the details of the operations. Second, the sudden evaporation occurs at noon, *after* all the specified operations have been completed. Thus, it must correspond to some additional, unspecified operation occurring precisely at noon (e.g., perhaps the urn is then turned upside-down, or the balls suddenly cease to exist). Furthermore, this new operation must be consistent with the continuity of the position functions of the balls. Since Earman & Norton affirm that at noon no balls are added or removed from the vase, such an additional operation seems to be ruled out.

In short, the argument for an empty noon-time urn must rely on more than just the continuity of the position functions of the balls. At heart there seems to be an underlying assumption that all the balls do in fact exit the urn before noon.

Since at noon an infinite number of consecutively numbered balls have been removed, the nth ball at the nth stage for an infinite number of stages until ball omega is removed at stage omega, it may seem that the urn must be empty.

However, the super-task as posed is completed when the last ball enters the urn, not when it exits. At noon balls numbered 1 to ω have been removed but the urn still contains balls numbered from $\omega + 1$ to 10ω. Perhaps this can be clarified by restating the problem slightly: if at stage n ball n is added to the urn and any balls numbered $n/10$ or less are removed, then at stage ω the urn contains all balls numbered higher than $\omega/10$. One could even postulate the existence of some mechanism that seals the urn once the last ball has been added to it, preventing any further exits. Hence, even though an infinite number of numbered balls have been removed from the urn, there still remains an infinite number of balls inside.

This is consistent with Earman & Norton's observation that $N(t)$ increases without limit for *all* instances before noon.

Furthermore, this conclusion is confirmed also by an examination of Allis & Koetsier's second super-task [1], where at the nth stage balls labeled $10n - 9$ to $10n - 1$ are added, whereas ball n, instead of being removed, is renumbered to $10n$. This leads to a number function $N(t)$ identical to that of the original super-task for all instances before noon. But now N is infinite at noon, each ball being labeled with a natural number followed by infinitely many zeros (an omega-sequence of zeros).

Allis & Koetsier exclaim that it is remarkable that the second super-task results in an infinite number function at noon, since at all times before noon it is identical to the number function of the first super-task. What is remarkable, however, is not that the second super-task results in a full urn, but that the first one should lead to an empty urn. It would seem that the balls left in the second urn at noon are precisely those that never exited the urn in the first super-task.

4. Subtracting Infinite Sets. The examination of Allis & Koetsier's second super-task supplies an answer also to the perplexing question: What is the label of the lowest numbered ball in the urn? At first sight this seems to refute the notion of a full urn at noon. If every specific natural number has been removed by noon, how can there be any numbers remaining in the urn?

On the basis of the Allis & Koetsiers' second super-task, one could simply reply that the lowest number in the urn is 1 followed by an infinite number of zeros.

Alternatively, one could argue that the set of natural numbers is of cardinality $\aleph_0$, encompassing $1, 2, 3, \dots \omega$. On the last action, just before noon, the last number to be removed from the urn is ω. Thus, the lowest number in the urn is the next number, $\omega + 1$.

One might ask, how can there be a next number $\omega + 1$ if ω is the largest natural number? The answer lies in the fact that an infinite set, unlike finite sets, can be partitioned into two or more subsets each having the same cardinality as the original set. Consequently, the subtraction of an infinite subset from an infinite set of the same cardinality does not necessarily leave one with the null set.

To illustrate this unique property of infinite sets, consider a further example. Suppose we have two urns, A and B. At the nth stage we place balls $10n - 9$ to $10n - 1$ in urn A, and ball $10n$ in urn B, renumbering the balls at each stage so that urn B contains balls numbered 1 to n, and urn A contains balls numbered $n + 1$ to $10n$. At each stage the balls in urn A will be identical to those in the urn of Littlewood's super-task. At noon urn B contains balls labeled $1, 2, \dots \omega$; urn A contains balls labeled $\omega + 1, \omega + 2, \dots \omega + \omega$, corresponding to the balls in Littlewood's urn at noon. This is equivalent to counting the naturals by first counting every tenth number and then the remaining numbers. (Suppose all the balls whose numbers are divisible by ten are red, the rest blue. Count first the red balls, then the blue balls. The red balls will then be labeled $1, 2, 3, \dots \omega$; the blue balls receive labels $\omega + 1, \omega + 2, \dots \omega + \omega$.)

Another approach is to consider the actions making up the super-task to be members of an infinite series

$$p1 + \cdots + p10 - p1 + p11 + \cdots + p20 - p2 + \cdots . \tag{2}$$

Littlewood [5] comments that, confronted with this series, "an analyst would at once observe that it was "null", and without noticing anything paradoxical".

Why would an analyst believe the infinite series to be "null"? Presumably because it might seem possible to rearrange it as

$$(p1 - p1) + (p2 - p2) + \cdots . \tag{3}$$

The latter series is clearly null, since we just add up an infinite number of zero's.

It is well-known, however, that a rearrangement of an infinite series leaves the sum unchanged if the series is absolutely convergent, but not necessarily in other cases. For example, a conditionally convergent infinite series may be rearranged to sum up to any desired extended real number. The dependence of the summation of an infinite series on its arrangement is noted also by Holgate [4].

To demonstrate this more concretely, we modify the super-task slightly in order to obtain a conditionally convergent infinite series. Let the balls vary in mass so that ball n has a mass of $1/n$ kg. Then we ask, what is the mass of the balls in the urn at noon? The super-task, broken down into individual actions, can be represented by the infinite series

$$(1 + \cdots + 1/10 - 1) + (1/11 + \cdots + 1/20 - 1/2) + \cdots . \tag{4}$$

One might think that, since any specific number added is eventually removed, this sum should be equivalent to the rearrangement

$$(1 - 1) + (1/2 - 1/2) + \cdots + (1/n - 1/n) + \cdots . \tag{5}$$

Yet these two series yield quite different sums. Series (4) sums to a value of approximately 2.3 kg (i.e., ln(10)); series (5), corresponding to Littlewood's reasoning for an empty urn, sums to zero. Again it is evident that the mere fact that any particular finite natural number added to the series is eventually removed is insufficient proof for the presumed emptiness of the urn at noon.

5. Conclusions. In this paper we have argued that a natural continuity condition to be applied to the Littlewood super-task is that of the continuity of the number $N(t)$ of balls in the urn, leading to the conclusion that the urn is full at noon. This condition need not contradict the continuity of position functions. It seems that the argument for an empty urn at noon requires some further elaboration as to what happens at noon. Finally, we have presented various examples demonstrating that the subtraction of one infinite series from another need not yield a null result.

Note. The editors encourage the submission of any response/discussion concerning this article.

References

1. L. V. Allis and T. Koetsier, "On Some Paradoxes of the Infinite," *British Journal for the Philosophy of Science*, 42 (1991), 187–194.
2. L. V. Allis and T. Koetsier, "On Some Paradoxes of the Infinite II," *British Journal for the Philosophy of Science*, 46 (1995), 235–247.
3. J. Earman and J. D. Norton, "Infinite Pains: The Trouble with Supertasks," *Paul Benacerrif: The Philosopher and His Critics*, S. Stich (ed.), Blackwell, New York, 1994.
4. P. Holgate, "Mathematical Notes on Ross's Paradox," *British Journal for the Philosophy of Science*, 45 (1994), 302–304.
5. J. E. Littlewood, *Littlewood's Miscellany*, B. Bollobas (ed.), Cambridge University Press, Cambridge, 1986.
6. S. Ross, *A First Course in Probability*, 3rd ed., Macmillan, New York & London, 1988.
7. J. P. Van Bendegem, "Ross' Paradox is an Impossible Super-Task," *British Journal for the Philosophy of Science*, 45 (1994), 743–748.

John Byl
Department of Mathematical Sciences
Trinity Western University
7600 Glover Road
Langley, B.C. V2Y 1Y1
Canada
email: byl@twu.ca