SOME REPRESENTATIONS OF $\zeta(3)$

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1. Introduction. The Riemann zeta function ζ is defined as

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z},$$

for each complex number z with real part Re $z > 1$. In this paper we only concentrate on $\zeta(3)$. R. Apéry [1] proved that $\zeta(3)$ is an irrational number using the formula

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}. \tag{1}$$

Motivated by Apéry's proof, F. Beukers [2] later gave a shorter proof of the irrationality of $\zeta(3)$ by means of double and triple integrals. Beukers' proof hinged on his formula

$$\zeta(3) = \int_0^1 \int_0^1 \frac{-\log xy}{1 - xy}\, dx\, dy \tag{2}$$

where the integrals can be justified by replacing $\int_0^1$ with $\lim\limits_{\varepsilon \to 0} \int_\varepsilon^{1-\varepsilon}$.

The value of $\zeta(3)$, however, remains unknown, let alone the values of ζ at other larger odd integers.

The formulas

$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} = \frac{2\pi\sqrt{3} + 9}{27},$$

$$\sum_{n=1}^{\infty} \frac{1}{n\binom{2n}{n}} = \frac{\pi\sqrt{3}}{9},$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}} = \frac{\pi^2}{18}$$

are easy to prove [3]. However, no one knows the value of

$$\sum_{n=1}^{\infty} \frac{1}{n^3\binom{2n}{n}}.$$

There is an interesting identity due to Comtet [4] that

$$\sum_{n=1}^{\infty} \frac{1}{n^4\binom{2n}{n}} = \frac{17\pi^4}{3240}$$

but there are no known values for

$$\sum_{n=1}^{\infty} \frac{1}{n^k\binom{2n}{n}}$$

for integers $k > 4$.

In section 2 we use Beukers' formula (2) to find a simple representation of $\zeta(3)$ in terms of a single integral instead of a double integral. In section 3 we obtain a series representation for $\zeta(3)$. The author hopes that some representation of $\zeta(3)$ in the literature can be used to evaluate $\zeta(3)$.

2. An Integral Representation of $\zeta(3)$. Let us write Beukers' formula as

$$\zeta(3) = \frac{1}{2}\int_0^1 \int_0^1 \frac{\log xy}{xy-1}\, dx\, dy$$

and consider $(x, y) \in (0,1) \times (0,1)$.

For a fixed y, substitute $w = xy - 1$ in the innermost integral. Then

$$\zeta(3) = \frac{1}{2}\int_0^1 \frac{1}{y}\int_{-1}^{y-1} \frac{\log(w+1)}{w}\,dw\,dy = \frac{1}{2}\int_0^1 \frac{1}{y}\lim_{a\to(-1)^+}\int_a^{y-1} \frac{\log(w+1)}{w}\,dw\,dy.$$

Now

$$\log(w+1) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{w^n}{n},\ \ |w| \leq 1.$$

Then

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{w^{n-1}}{n}$$

converges uniformly to $\frac{\log(w+1)}{w}$ on $[a, y-1]$.

So

$$\zeta(3) = \frac{1}{2}\int_0^1 \frac{1}{y}\lim_{a\to(-1)^+}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_a^{y-1} w^{n-1}\,dwdy$$

$$= \frac{1}{2}\int_0^1 \frac{1}{y}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left\{\frac{(y-1)^n}{n} - \frac{(-1)^n}{n}\right\}dy$$

$$= \frac{1}{2}\int_0^1 \frac{1}{y}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}(y-1)^n}{n^2} + \frac{1}{n^2}\right\}dy$$

$$= \frac{1}{2}\int_0^1 \frac{1}{y}\left\{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(y-1)^n}{n^2} + \frac{\pi^2}{6}\right\}dy$$

since $y-1 \in (0,1)$ and absolute convergence implies convergence. Using the functional equation for the dilogarithm

$$\sum_{n=1}^{\infty} \frac{y^n}{n^2}$$

[3] we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(y-1)^n}{n^2} = -\sum_{n=1}^{\infty} \frac{(1-y)^n}{n^2}$$

$$= \log(1-y)\log y + \sum_{n=1}^{\infty} \frac{y^n}{n^2} - \frac{\pi^2}{6}$$

so

$$\zeta(3) = \frac{1}{2}\int_0^1 \frac{1}{y}\left\{\log(1-y)\log y + \sum_{n=1}^{\infty}\frac{y^n}{n^2}\right\} dy$$

$$= \frac{1}{2}\int_0^1 \frac{\log(1-y)\log y}{y}\, dy + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^3}$$

or

$$\zeta(3) = \int_0^1 \frac{\log(1-y)\log y}{y}\, dy.$$

It is worth mentioning that, by making a simple change of variable, the above integral representation can be written as

$$\zeta(3) = \int_0^1 \frac{\log x}{x-1} \log \frac{1}{1-x}\, dx$$

where it is easy to see that

$$\int_0^1 \frac{\log x}{x-1}\, dx = \frac{\pi^2}{6} = \zeta(2).$$

3. A Series Representation of $\zeta(3)$. Using the well-known formula [3]

$$2(\sin^{-1} x)^2 = \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n^2\binom{2n}{n}}$$

we have

$$2\int_0^{1/2} (\sin^{-1} y)^2 \frac{dy}{y} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^3\binom{2n}{n}}.$$

On the other hand a simple integration by substitution followed by integration by parts yields

$$2\int_0^{1/2} (\sin^{-1} y)^2 \frac{dy}{y} = -\int_0^{\pi/3} x\log(2\sin\tfrac{1}{2}x)dx.$$

Combining we get

$$\sum_{n=1}^{\infty} \frac{1}{n^3\binom{2n}{n}} = -2\int_0^{\pi/3} x\log(2\sin\tfrac{1}{2}x)dx. \tag{3}$$

Now clearly

$$\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}, \quad |z| < 1$$

even at boundary points except for $z = 1$, i.e. except at the points $z = e^{ix}$ with $x \neq 2k\pi$. Consider the interval $(0, 2\pi)$. Now

$$\begin{aligned} 1 - z = 1 - e^{ix} &= (1 - \cos x) - \sin x i \\ &= 2\sin^2 \tfrac{x}{2} - 2\sin\tfrac{x}{2}\cos\tfrac{x}{2} i \\ &= 2\sin\frac{x}{2}\left(\sin\frac{x}{2} - \cos\frac{x}{2} i\right) \\ &= 2\sin\frac{x}{2}\left\{\cos\left(-\frac{\pi}{2} + \frac{x}{2}\right) + \sin\left(-\frac{\pi}{2} + \frac{x}{2}\right) i\right\} \\ &= 2\sin\tfrac{x}{2} e^{(-\frac{\pi}{2} + \frac{x}{2})i}. \end{aligned}$$

So

$$\log(1 - z) = \log(2\sin\frac{x}{2}) + (-\frac{\pi}{2} + \frac{x}{2})i.$$

Applying Abel's theorem for trigonometric series we get

$$\log\left(2\sin\frac{x}{2}\right) = -\sum_{n=1}^{\infty}\frac{\cos nx}{n}, \quad x \in (0, 2\pi).$$

Using formula (3) we can now write

$$\sum_{n=1}^{\infty}\frac{1}{n^3\binom{2n}{n}} = 2\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\pi/3} x\cos nx\, dx.$$

Integrating by parts twice we get

$$\sum_{n=1}^{\infty}\frac{1}{n^3\binom{2n}{n}} = \frac{2\pi}{3}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{3}}{n^2} + 2\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi}{3}}{n^3} - 2\zeta(3).$$

So

$$\zeta(3) = \frac{\pi}{3} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{3}}{n^2} + \sum_{n=1}^{\infty} \frac{\cos \frac{n\pi}{3}}{n^3} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}}.$$

Since the middle term is $\frac{1}{3}\zeta(3)$ [5], we consequently have the following series representation

$$\zeta(3) = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{3}}{n^2} - \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}}.$$

References

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