A NOTE ON THE SUM $\sum 1/w_{k2^n}$

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1. Historical Results. In 1974, Millin [13] published a problem stating that

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}. \tag{1}$$

This spurred a flurry of activity: [1, 3, 4, 5, 6, 7, 8, 17]. Most investigators, however, overlooked the fact that Lucas studied such sums back in 1878. He showed in [11], equation (125), that if $k \neq 0$, then

$$\sum_{n=1}^{N} \frac{Q^{k2^{n-1}}}{u_{k2^n}} = \frac{Q^k u_{k(2^N-1)}}{u_k u_{k2^N}} \tag{2}$$

where u_n is a second order linear recurrence defined by

$$u_{n+2} = Pu_{n+1} - Qu_n, \quad u_0 = 0, \quad u_1 = 1.$$

If we use the identity $Q^{n-1}u_{m-n} = u_n u_{m-1} - u_m u_{n-1}$, we can express formula (2) in the form

$$\sum_{n=1}^{N} \frac{Q^{k2^{n-1}}}{u_{k2^n}} = Q\left[\frac{u_{k2^N-1}}{u_{k2^N}} - \frac{u_{k-1}}{u_k}\right]. \tag{3}$$

If $Q = -1$, as is the case for Fibonacci, Lucas, and Pell numbers, then equation (3) becomes

$$\sum_{n=0}^{N} \frac{1}{u_{k2^n}} = \frac{1+u_{k-1}}{u_k} + \frac{1-(-1)^k}{u_{2k}} - \frac{u_{k2^N-1}}{u_{k2^N}} \tag{4}$$

where we have handled the terms when n is 0 and 1 specially. For all subsequent terms, the exponent of Q is even and hence the numerator is 1. An equivalent formula found by Greig [6] is

$$\sum_{n=0}^{N} \frac{1}{u_{k2^n}} = \frac{1}{u_k} + \frac{1+u_{2k-1}}{u_{2k}} - \frac{u_{k2^N-1}}{u_{k2^N}}. \tag{5}$$

When $< u_n >$ is the Fibonacci sequence, equation (4) becomes the result found by Greig in [5]. Hoggatt and Bicknell [8] found an equivalent result, expressing their answer in terms of Fibonacci and Lucas numbers. This generalized the result they gave in [7]. Brady [2] found an equivalent result, expressing his answer in terms of the golden ratio. When $< u_n >$ is the Pell sequence, equation (4) becomes the result found by Horadam [10]. In equation (3), if we let $Q = 1$, we get the results found by Melham and Shannon [12].

Lucas [11] also found that if $k \neq 0$ and $p \neq 0$, then

$$\sum_{n=0}^{N} \frac{Q^{kp^n} u_{k(p-1)p^n}}{u_{kp^n} u_{kp^{n+1}}} = \frac{Q^k u_{k(p^{N+1}-1)}}{u_k u_{kp^{N+1}}}. \tag{6}$$

This, again, was overlooked by later researchers. Formula (6) is equivalent to equation (6) of Bruckman and Good [3]. If we let $P = x$ and $Q = -1$, then we get a result found by Popov [16], equation (4), for the Fibonacci polynomials. This, in turn, generalizes results for Fibonacci numbers found by Bergum and Hoggatt [1]. Brady [2] found an equivalent result for Fibonacci numbers, expressing his answer in terms of the golden ratio.

2. New Results. Instead of the sequence $< u_n >$, we can study the sequence $< w_n >$ defined by

$$w_{n+2} = Pw_{n+1} - Qw_n, \quad w_0,\ w_1 \text{ arbitrary.}$$

In order that no denominator be 0, we will make the assumption that $w_n \neq 0$ for $n > 0$. We also assume that k is a fixed positive integer and that $P^2 \neq 4Q$. Finally, we let

$$\alpha = \frac{P + \sqrt{P^2 - 4Q}}{2} \quad \text{and} \quad \beta = \frac{P - \sqrt{P^2 - 4Q}}{2}$$

and note that $\alpha\beta = Q$.

In [10], a formula for $\sum 1/w_{k2^n}$ is claimed to be found for the case where $Q = -1$. However, this formula is not correct unless $w_0 = 0$. For $k = 1$, the supposed formula is

$$\sum_{n=0}^{N} \frac{1}{w_{2^n}} = \frac{1}{w_1} + \frac{1 + w_1}{w_2} - \frac{w_{2^N - 1}}{w_{2^N}}.$$

A counterexample to this claim is the Lucas sequence with $N = 2$. Perhaps the author inadvertently omitted the hypothesis $w_0 = 0$, in which case the above formula and the formulas given on page 112 of [10] are valid. These results are then a special case of the following.

Theorem 1. If $w_0 = 0$, then

$$\sum_{n=1}^{N} \frac{Q^{k2^{n-1}}}{w_{k2^n}} = \frac{Q^k w_{k(2^N - 1)}}{w_k w_{k2^N}}. \tag{7}$$

Proof. We use the identity $w_n = w_1 u_n - Q w_0 u_{n-1}$ which comes from [9]. Letting $w_0 = 0$, we find that $w_n = w_1 u_n$ for all n. Substituting $u_n = w_n / w_1$ in equation (2) gives us the desired result.

Corollary 1. If $w_0 = 0$ and $Q = 1$, then

$$\sum_{n=1}^{N} \frac{1}{w_{k2^n}} = \frac{w_{k(2^N - 1)}}{w_k w_{k2^N}} = w_1 \left[\frac{w_{k2^N - 1}}{w_{k2^N}} - \frac{w_{k-1}}{w_k} \right]. \tag{8}$$

Corollary 2. If $w_0 = 0$ and $Q = -1$, then

$$\sum_{n=1}^{N} \frac{1}{w_{k2^n}} = \frac{1-(-1)^k}{w_{2k}} + \frac{w_{k(2^N-1)}}{w_k w_{k2^N}} = \frac{1+w_{2k-1}}{w_{2k}} - \frac{w_{k2^N-1}}{w_{k2^N}}. \tag{9}$$

In a similar manner, formula (6) continues to hold when u is replaced by w, provided that $w_0 = 0$.

Sums to infinity can also be obtained by letting $N \to \infty$ in any of the above formulas. We use the following fact, which is taken from [15].

Lemma. For all integers r,

$$\lim_{N\to\infty} \frac{u_{N-r}}{u_N} = \begin{cases} \alpha^r, & \text{if } |\beta/\alpha| < 1, \\ \beta^r, & \text{if } |\beta/\alpha| > 1. \end{cases}$$

When $w_0 = 0$, so that w_n is proportional to u_n, we may replace u by w in the above lemma. Letting $N \to \infty$ in formula (7) and recalling that $\alpha\beta = Q$, we get the following.

Theorem 2. If $w_0 = 0$, then

$$\sum_{n=1}^{\infty} \frac{Q^{k2^{n-1}}}{w_{k2^n}} = \begin{cases} \beta^k/w_k, & \text{if } |\beta/\alpha| < 1, \\ \alpha^k/w_k, & \text{if } |\beta/\alpha| > 1. \end{cases} \tag{10}$$

If $< w_n >$ is the Fibonacci sequence, then formula (10) reduces to formula (1), and this agrees with the value found by Lucas in 1878: formula (127) of [11].

References

1. G. E. Bergum and V. E. Hoggatt, Jr., "Infinite Series with Fibonacci and Lucas Polynomials," *The Fibonacci Quarterly*, 17 (1979), 147–151.
2. W. G. Brady, "Additions to the Summation of Reciprocal Fibonacci and Lucas Series," *The Fibonacci Quarterly*, 9 (1971), 402–404, 412.
3. P. S. Bruckman and I. J. Good, "A Generalization of a Series of De Morgan with Applications of Fibonacci Type," *The Fibonacci Quarterly*, 14 (1976), 193–196.
4. I. J. Good, "A Reciprocal Series of Fibonacci Numbers," *The Fibonacci Quarterly*, 12 (1974), 346.
5. W. E. Greig, "Sums of Fibonacci Reciprocals," *The Fibonacci Quarterly*, 15 (1977), 46–48.
6. W. E. Greig, "On Sums of Fibonacci-Type Reciprocals," *The Fibonacci Quarterly*, 15 (1977), 356–358.
7. V. E. Hoggatt, Jr. and M. Bicknell, "A Primer for the Fibonacci Numbers, Part XV: Variations on Summing a Series of Reciprocals of Fibonacci Numbers," *The Fibonacci Quarterly*, 14 (1976), 272–276.
8. V. E. Hoggatt, Jr. and M. Bicknell, "A Reciprocal Series of Fibonacci Numbers with Subscripts $2^n k$," *The Fibonacci Quarterly*, 14 (1976), 453–455.
9. A. F. Horadam, "Basic Properties of a Certain Generalized Sequence of Numbers," *The Fibonacci Quarterly*, 3 (1965), 161–176.
10. A. F. Horadam, "Elliptic Functions and Lambert Series in the Summation of Reciprocals in Certain Recurrence-Generated Sequences," *The Fibonacci Quarterly*, 26 (1988), 98–114.
11. E. Lucas, "Théorie des Fonctions Numériques Simplement Périodiques," *American Journal of Mathematics*, 1 (1878), 184–240, 289–321.
12. R. S. Melham and A. G. Shannon, "On Reciprocal Sums of Chebyshev Related Sequences," *The Fibonacci Quarterly*, 33 (1995), 194–202.
13. D. A. Millin, "Problem H-237," *The Fibonacci Quarterly*, 12 (1974), 309.
14. B. S. Popov, "On Certain Series of Reciprocals of Fibonacci Numbers," *The Fibonacci Quarterly*, 22 (1984), 261–265.
15. B. S. Popov, "Summation of Reciprocal Series of Numerical Functions of Second Order," *The Fibonacci Quarterly*, 24 (1986), 17–21.

16. B. S. Popov, "A Note on the Sums of Fibonacci and Lucas Polynomials," *The Fibonacci Quarterly*, 23 (1985), 238–239.

17. A. G. Shannon, "Solution to Problem H-237: Sum Reciprocal!," *The Fibonacci Quarterly*, 14 (1976), 186–187.

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