ISOSCELES TRIANGLES EQUAL IN PERIMETER AND AREA

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Abstract. Beginning with the observation that to every non-equilateral, isosceles triangle, there is a unique, noncongruent isosceles triangle with the same perimeter and the same area, we determine all such pairs with integer sides and integer areas. We also determine those pairs in which the sides in each isosceles triangle are relatively prime. Finally, we give an infinite family of pairs of triangles equal in perimeter and in area, one isosceles and one non-isosceles.

1. Isaac Newton [4] gave an elegant solution to the so-called Roberval problem (also called van Schooten's Problem): Find two (rational) isosceles triangles which shall be equal in area and perimeter.

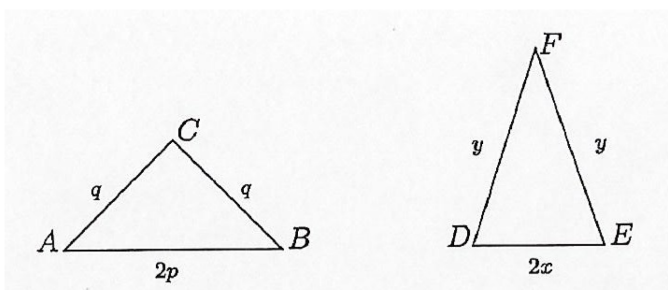

Figure 1. Isosceles Triangles Equal in Perimeter and Area.

Let the triangles be ABC, DEF; their bases $AB = 2p$, $DE = 2x$; their slant sides $AC = q$, $DF = y$. Then $p + q = x + y$ and $p\sqrt{q^2 - p^2} = x\sqrt{y^2 - x^2}$, that is $p^2q^2 - p^4 = x^2y^2 - x^4$. Divide by the equal quantities, $p + q$ and $x + y$ and so $p^2q - p^3 = x^2y - x^3$. In place of y write $p + q - x$. It follows that

$$p^2q - p^3 = px^2 + qx^2 - 2x^3 \text{ and so } q = \frac{p^3 + px^2 - 2x^3}{p^2 - x^2}.$$

Divide through by $p-x$ to obtain $q=\frac{p^2+px+2x^2}{p+x}$.

Since this solution gives q in terms of p and x, it does not construct an isosceles triangle equal in perimeter and area to a given one. The last expression for q, however, can be construed as an equation in x:

$$2x^2-(q-p)x-p(q-p)=0.$$

Since $q>p$, it is clear that this quadratic equation has exactly one positive root given by

$$x=\frac{1}{4}\left((q-p)+\sqrt{(q-p)(q+7p)}\right). \tag{1}$$

Note that $y=p+q-x$ is always positive since $\sqrt{(q-p)(q+7p)}<q+3p$. In (1), $x=p$ if and only if $q=2p$, i.e., the given triangle is equilateral. We conclude, therefore, that to every non-equilateral, isosceles triangle, there corresponds a unique, non-congruent isosceles triangle equal in perimeter and area.

2. We shall construct pairs of such isosceles triangles with integer sides. Suppose then, that in (1) above, p and q are integers. The base $2x$ of the second isosceles triangle is an integer if and only if $\frac{q-p}{q+7p}=r^2$ for a rational number r. It follows that $\frac{p}{q}=\frac{1-r^2}{1+7r^2}$ and $r<1$. Writing $r=\frac{k}{h}$ for integers $h>k$, and clearing denominators, we obtain the following pair.

Table 1. Pair of Integer Isosceles Triangles Equal in Perimeter and Area.

triangle	$\triangle_1$	$\triangle_2$
legs	h^2+7k^2	$2(h^2-hk+2k^2)$
base	$2(h^2-k^2)$	$4k(h+k)$
height	$4k\sqrt{h^2+3k^2}$	$2(h-k)\sqrt{h^2+3k^2}$
perimeter	$4(h^2+3k^2)$	$4(h^2+3k^2)$
area	$4k(h^2-k^2)\sqrt{h^2+3k^2}$	$4k(h^2-k^2)\sqrt{h^2+3k^2}$

3. The common area of the pair of isosceles triangles in Table 1 above is, in general, not an integer. To make these triangles Heronian, i.e., with sides and areas being all integers, it is enough to choose h and k so that $h^2 + 3k^2$ is the square of an integer. This is the case if and only if

$$\frac{h}{k} = \frac{m^2 - 3n^2}{2mn}$$

for relatively prime integers m and n, with $\gcd(m, 3) = 1$. See, for example, [2]. With $h = m^2 - 3n^2$ and $k = 2mn$, $(m > 3n)$, we obtain the following pair of isosceles triangles.

Table 2. Pair of Heronian Isosceles Triangles Equal in Perimeter and Area.

triangle	$\triangle_1$	$\triangle_2$
legs	$m^4 + 22m^2n^2 + 9n^4$	$2(m^4 - 2m^3n + 2m^2n^2 + 6mn^3 + 9n^4)$
base	$2(m^2 - n^2)(m^2 - 9n^2)$	$8mn(m - n)(m + 3n)$
height	$8mn(m^2 + 3n^2)$	$2(m + n)(m - 3n)(m^2 + 3n^2)$
perimeter	$4(m^2 + 3n^2)^2$	$4(m^2 + 3n^2)^2$
area	$8mn(m^2 - n^2)(m^2 - 9n^2)(m^2 + 3n^2)$	$8mn(m^2 - n^2)(m^2 - 9n^2)(m^2 + 3n^2)$
parameters	$(m^2 + 3n^2, 4mn)$	$(m(m - n), n(m + 3n))$
mag. factor	1	2

4. Each isosceles triangle in Table 2 is obtained by juxtaposing two congruent Pythagorean triangles with parameters (u, v) and magnification factor μ along the sides of length $\mu(u^2 - v^2)$, so that the base and the legs have lengths $\mu \cdot 4uv$ and $\mu(u^2 + v^2)$, respectively. The gcd's of the two pairs of parameters can be computed easily by using the Euclidean algorithm, and are as follows.

$$\begin{aligned} g_1 &:= \gcd(m^2 + 3n^2, 4mn) = \gcd(m - n, 2)^2, \\ g_2 &:= \gcd(m(m - n), n(m + 3n)) = \gcd(m - n, 4). \end{aligned}$$

Here, we recall that $\gcd(m, 3) = 1$. Note that $(\frac{m^2+3n^2}{g_1}, \frac{4mn}{g_1})$ are both odd, if m and n are both odd. This means that the sides of the first isosceles triangle have gcd

$$d_1 = \gcd(m - n, 2)^5.$$

Similarly, the sides of the second isosceles triangle have gcd

$$d_2 = 4 \cdot \gcd(m - n, 2) \cdot \gcd(m - n, 4).$$

Therefore, by dividing the sides of the two triangles by

$$d := \gcd(d_1, d_2) = \gcd(m-n, 2)^3 \cdot \gcd(m-n, 4),$$

we obtain a pair of isosceles triangles in which the lengths of the bases and the legs have no common divisor. These two triangles are both primitive when $d_1 = d_2$. This is the case if and only if $\gcd(m-n, 2)^4 = 4 \cdot \gcd(m-n, 4)$, or equivalently, $m \equiv n \pmod 4$. Indeed, if $\gcd(m, n) = 1$ and $m \equiv n \pmod 4$, then they are both odd and the parameters for the Pythagorean component of $\triangle_1$, namely $\frac{m^2+3n^2}{g_1}$ and $\frac{4mn}{g_1}$, are both odd. As noted above, the primitive Pythagorean triangle obtained by reducing $\triangle_1$ is given by parameters

$$\frac{1}{2g_1}(m^2 + 3n^2 \pm 4mn) = \frac{1}{8}(m \pm n)(m \pm 3n).$$

The case for the primitive Pythagorean triangle reduced from $\triangle_2$ is similar. We summarize this in the following theorem.

Theorem 1. Every pair of primitive, isosceles Heronian triangles equal in perimeter and area has its Pythagorean components given by parameters

(i) $(\frac{1}{8}(m+n)(m+3n), \frac{1}{8}(m-n)(m-3n))$,

(ii) $(\frac{1}{4} \cdot m(m-n), \frac{1}{4} \cdot n(m+3n))$, where m and n are relatively prime integers such that $m > 3n$, $\gcd(m, 3) = 1$, and $m \equiv n \pmod 4$.

The following table contains a list of pairs of primitive, isosceles Heronian triangles equal in perimeter and in area, arranged in increasing magnitudes of $m+n$. We need only consider (m, n) specified in Theorem 1.

Table 3. Pairs of Primitive Isosceles Heronian Triangles Equal in Perimeter and Area.

(m, n)	$(2p, q, q)$	$(2x, y, y)$	perim.	area
$(5, 1)$	$(24, 37, 37)$	$(40, 29, 29)$	98	420
$(11, 3)$	$(280, 1229, 1229)$	$(1320, 709, 709)$	2738	170940
$(13, 1)$	$(1680, 1009, 1009)$	$(624, 1537, 1537)$	3698	469560
$(17, 1)$	$(5040, 2809, 2809)$	$(1360, 4649, 4649)$	10658	3127320
$(17, 5)$	$(1056, 7753, 7753)$	$(8160, 4201, 4201)$	16562	4084080
$(19, 3)$	$(6160, 6329, 6329)$	$(6384, 6217, 6217)$	18818	17029320
$(23, 3)$	$(14560, 12041, 12041)$	$(11040, 13801, 13801)$	38642	69822480
$(25, 1)$	$(24024, 12637, 12637)$	$(4200, 22549, 22549)$	49298	47147100
$(23, 7)$	$(2640, 27241, 27241)$	$(28336, 14393, 14393)$	57122	35915880
$(29, 1)$	$(43680, 22681, 22681)$	$(6496, 41273, 41273)$	89042	133638960

5. Given an isosceles Heronian triangle, it is sometimes possible to find a non-isosceles Heronian triangle equal in perimeter and area. The smallest example of such a pair is

$$(29, 29, 40) \text{ and } (25, 34, 39).$$

According to Table 3 above, the isosceles triangle can also be replaced by $(37, 37, 24)$. We give an infinite family of such pairs of triangles below.

Let h and k be relatively prime positive integers. The two triangles with sides

$$\begin{aligned} a_1 &= 8(h^2 - hk + k^2)(h^2 - hk + 4k^2), \\ b_1 = c_1 &= 5h^4 - 10h^3k + 21h^2k^2 - 16hk^3 + 20k^4; \end{aligned}$$

$$\begin{aligned} a_2 &= 5h^4 - 22h^3k + 45h^2k^2 - 40hk^3 + 32k^4, \\ b_2 &= 5h^4 + 2h^3k + 9h^2k^2 - 4hk^3 + 20k^4, \\ c_2 &= 4(h^2 - hk + k^2)(2h^2 - 2hk + 5k^2) \end{aligned}$$

have a common perimeter $18(h^2 - hk + 2k^2)^2$ and a common area

$$12|(h + k)(h - 2k)|(h^2 - hk + k^2)(h^2 - hk + 2k^2)(h^2 - hk + 4k^2).$$

The second triangle is non-isosceles provided $(h, k) \neq (1, 1), (1, 2)$. Here are examples of pairs of triangles obtained from small values of h and k. In each case, we have divided the sides of the triangles by their greatest common divisor.

Table 4. Pairs of Heronian Triangles, One Isosceles and
One Non-Isosceles, Equal in Perimeter and Area.

(h,k)	(a_1,b_1,c_1)	(a_2,b_2,c_2)	$(s,\triangle)$
$(1,3)$	$(476,338,338)$	$(464,401,287)$	$(576,57120)$
$(3,1)$	$(140,74,74)$	$(32,137,119)$	$(144,1680)$
$(1,4)$	$(6344,4397,4397)$	$(6269,5021,3848)$	$(7569,9658740)$
$(2,3)$	$(476,338,338)$	$(401,464,287)$	$(576,57120)$
$(3,2)$	$(1064,557,557)$	$(389,1061,728)$	$(1089,87780)$
$(4,1)$	$(416,233,233)$	$(116,389,377)$	$(441,21840)$
$(1,5)$	$(448,305,305)$	$(445,340,273)$	$(529,46368)$
$(5,1)$	$(112,65,65)$	$(37,100,105)$	$(121,1848)$

Further examples of infinite families of pairs of Heronian triangles equal in perimeter and area will be studied in another paper [3].

References

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