A COMPOSITION PROBLEM INVOLVING ANALYTIC FUNCTIONS

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1. Introduction. One of the beautiful principles of Fourier analysis maintains that when an analytic function ϕ acts upon a function f continuous on the unit circle T in the complex plane, the composition $\phi(f)$ frequently inherits "nice" properties possessed by f. An example of this idea is the Wiener-Levy Theorem, which asserts that if f has an absolutely convergent Fourier series and if ϕ is a function analytic in a neighborhood of $f(T)$, then the composition $\phi(f)$ has an absolutely convergent Fourier series as well [4].

In this note we consider a composition problem in which the "nice" property can be expressed in terms of analyticity or vanishing Fourier coefficients. Specifically, we let $A(D)$ (sometimes referred to as the "disk algebra") denote the algebra of functions that are continuous on T and that possess analytic extensions into the open unit disk D. Equivalently, owing to the Poisson integral, a function f belongs to $A(D)$ if and only if it is continuous on T and has the property that its Fourier coefficients

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt$$

are equal to zero for all $n < 0$.

Clearly if f belongs to $A(D)$ and if φ is analytic in a neighborhood of $f(\overline{D})$, then $g = \varphi(f)$ belongs to $A(D)$, and one may express its extension in D as $g(z) = \varphi(f(z))$. Moreover, the elementary examples consisting of the pairs $\varphi(z) = z^2$, $f(z) = \sqrt{z}$, and $\varphi(z) = e^z$, $f(z) = \log(z)$ (where any branch of f is chosen) illustrate how an analytic function φ can map a function f not in $A(D)$ to a function $g = \varphi(f)$ back in this class.

Yet for neither of these two examples is the function f continuous on the unit circle. A more difficult question, one raised by Forelli [2], arises when one asks whether, for such an arbitrary function φ, there exists a function f continuous on T and not in $A(D)$ that is mapped to a function $g = \varphi(f)$ in $A(D)$. In terms of analyticity, $g = \varphi(f)$ has an analytic extension $g(z)$ in D even though f, while continuous on T, does not possess such an extension. In terms of Fourier coefficients,

f and g are continuous functions with the properties $\hat{g}(n) = \widehat{\varphi(f)}(n) = 0$ for all $n < 0$ yet $\hat{f}(n) \neq 0$ for at least one $n < 0$. We show that the answer to this question depends upon the choice of φ. Interestingly, the answer differs for the two choices $\varphi(z) = z^2$ and $\varphi(z) = e^z$.

2. Results. We begin with a theorem that uses classical complex analysis to provide the answer for a class of functions that includes $\varphi(z) = z^2$.

Theorem 1. Let $\varphi(z)$ be analytic in some region X, where its derivative has at least one zero. Then there exists a function f that is continuous on T, that does not belong to $A(D)$, that satisfies $f(T) \subseteq X$, and that is mapped by φ to a function $g = \varphi(f)$ in $A(D)$.

Proof. Suppose that $\varphi'(\alpha) = 0$ and $\varphi(\alpha) = r$. By considering the function $\varphi(z+\alpha) - r$, we see that it suffices to prove the theorem for the case when X contains the origin and φ has a zero of order $k \geq 2$ at the origin. For $|w|$ less than some fixed positive δ, a classical result sometimes known as Weierstrass' *Vorbereitungssatz* [3], states that the equation $\varphi(z) = w$ has k roots $z_1(w), \ldots, z_k(w)$ of the form $z_i(w) = g(w^{1/k})$, where $g(\zeta)$ is a function analytic in some neighborhood of the origin and where the k-determinations of $w^{1/k}$ yield the different branches of $z_1(w), \ldots, z_k(w)$. In particular, each $z_j(w)$ is analytic in $D(0;\delta)\backslash(-\infty, 0]$ and is continuous and equal to 0 at $w = 0$.

Now choose $\lambda > 0$ with the property that $2\lambda^2 < \delta$. Under the map $w = \lambda^2(\zeta^2 + 1)$ the disc $|\zeta| < 1$ is mapped to the disc $|w - \lambda^2| < \lambda^2$ so that the functions $z_j(\lambda^2(\zeta^2 + 1))$ belong to $A(D)$ and are all equal to 0 at the points $\pm i$. Let

$$f(e^{it}) = \begin{cases} z_1(\lambda^2(e^{2it} + 1)), & \text{if } \mathbb{R}(e^{it}) \geq 0 \\ z_2(\lambda^2(e^{2it} + 1)), & \text{if } \mathbb{R}(e^{it}) < 0. \end{cases}$$

Then f is continuous on T, has its range contained in X, and is mapped by φ to

$$\varphi(f(e^{it})) = \varphi(z_j(\lambda^2(e^{2it} + 1))) = \lambda^2(e^{2it} + 1) \in A(D).$$

However, if f itself belongs to $A(D)$, then so do the differences $f - z_1(\lambda^2(e^{2it} + 1))$ and $f - z_2(\lambda^2(e^{2it} + 1))$. Since the nullset on T of any nonzero function in $A(D)$ has Lebesgue measure zero [4], it follows that $z_1 \equiv z_2$ in $D(0;\delta)\backslash(-\infty, 0]$, a fact that contradicts the manner in which these roots were chosen. Thus, f itself does not belong to $A(D)$.

Ideally, proving that a function φ with nonvanishing derivative is capable of mapping a continuous function f to $g = \varphi(f)$ in $A(D)$ only if f itself belongs to $A(D)$ would provide a complete answer to Forelli's question. The pair $\varphi(z) = \frac{1}{z}$, $f(e^{it}) = e^{-it}$ illustrates, however, that this is not the case.

Yet, this example distinguishes itself in that there exists no simply connected region containing $f(T)$ yet not containing zero, the exceptional value of φ. The following theorem illustrates that it is precisely this fact that prevents f from belonging to $A(D)$.

We state this theorem using a generalization of the idea of exceptional value, known as the asymptotic value. The function φ analytic in the region X is said to have an asymptotic value at the complex number α if there exists a continuous mapping z from $[0, 1)$ into X with the properties that $\varphi(z(t)) \to \alpha$ and $z(t) \to B(X)$, the ideal boundary of X in the extended complex plane, as $t \to 1$. It is a well-known fact that if an analytic function φ has a nonvanishing derivative, the only possible singularities of a branch of its inverse occur at asymptotic values [1].

Theorem 2. Suppose φ is analytic in some region X where its derivative is nonvanishing. Let f be a continuous function on T for which $f(T) \subseteq X$ and $g = \varphi(f)$ belongs to $A(D)$. If there exists a simply connected region Ω satisfying $f(T) \subseteq \Omega \subseteq X$ and having the property that $\varphi(\Omega)$ contains no asymptotic values of φ, then f itself belongs to $A(D)$.

Proof. We begin with the claim $g(\overline{D}) \subseteq \varphi(\Omega)$. If this inclusion does not hold, then there exists some β in $g(D)$ with the property that $\varphi - \beta$ is zero-free on Ω. Hence, $\varphi - \beta$ possesses an analytic logarithm G on Ω. If we let $\mathrm{Ind}_g(\beta)$ denote the usual winding number of the curve $g(T)$ with respect to β and if we note that Ω is simply connected, we conclude that

$$\mathrm{Ind}_g(\beta) = \mathrm{Ind}_{\varphi(f)}(\beta) = \mathrm{Ind}_{e^{G(f)}}(0) = 0.$$

But the Argument Principle dictates that $\mathrm{Ind}_g(\beta)$ also represents the number of zeros in D of $g - \beta$. From this contradiction, the claim follows.

Proceeding with the proof, we choose a function element ψ analytic in a neighborhood of $g(1)$ with the property that $\psi(g(1)) = f(1)$. The element $\psi(g)$ then analytically continues along all arcs in D that start at $z = 1$, since $g(\overline{D})$ contains no asymptotic values of φ. By the Monodromy Theorem, $\psi(g)$ defines an analytic function in D whose extension to T agrees with f. Hence, f belongs to $A(D)$.

3. Discussion. The hypothesis of the preceding theorem is always satisfied if the function φ is analytic in some simply connected region X, has a nonvanishing derivative there, and possesses no asymptotic values. The simplest example of such a function is of course $\varphi(z) = e^z$. Another example includes the so-called modular function [5].

While these results answer Forelli's question in large part, they do leave situations unaccounted for. Such situations occur when one assumes in the preceding hypothesis that X is simply connected and drops altogether the condition regarding asymptotic values. An interesting example of a function φ that leads to such a situation, one for which the author does not know the outcome to the original question, is $\varphi(z) = \int_{w=0}^{w=z} e^{w^2} dw$.

References

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