A NOTE ON LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Richard E. Bayne, James E. Joseph, Myung H. Kwack, and Thomas H. Lawson

As is well-known, when $a \neq b$,

$$y = c_1 e^{ax} + c_2 e^{bx} \tag{1}$$

is a general solution of the differential equation

$$y'' - (a+b)y' + aby = 0. \tag{2}$$

In [1], Euler points out the fact that

$$y = c_1 e^{ax} + c_2 x e^{ax} \tag{3}$$

is a general solution of the differential equation

$$y'' - 2ay' + a^2 y = 0, \tag{4}$$

although easily checked, is not so easily motivated. He provided a technique to show how this fact arises. In this paper, we offer a technique which shows how solutions (1) and (3) arise as general solutions of (2) and (4), respectively. We then use this technique, along with induction, to give the form of a general solution of the nth order linear differential equation with complex coefficients.

Motivation for Technique. If a and b are complex constants, a general solution of the differential equation

$$y'' - (a+b)y' + aby = q(x) \tag{5}$$

is

$$y = e^{ax} \int e^{(b-a)x} \left(\int e^{-bx} q(x) dx \right) dx. \tag{6}$$

Proof. Writing the equivalent form

$$(y' - ay)' - b(y' - ay) = q(x) \tag{7}$$

for the differential equation in (5) and multiplying through by the integrating factor e^{-bx}, we find that

$$y' - ay = e^{bx} \int e^{-bx} q(x)dx \tag{8}$$

is a general solution of (7). Multiplying through by the integrating factor e^{-ax} in (8), we obtain

$$y = e^{ax} \int e^{(b-a)x} \left(\int e^{-bx} q(x)dx \right) dx \tag{9}$$

as a general solution of (5).

Utilization of the same technique and induction yields the following theorem.

Theorem. If n is a positive integer, $a_1, \dots, a_n$ are complex constants, $s_0 = 1$, and for $k = 1, \dots, n$, $s_k = \sum_\phi \prod a_{\phi(j)}$, where ϕ is a strictly increasing function from $\{1, \dots, k\}$ to $\{1, \dots, n\}$, then a general solution to

$$\sum_{k=0}^{n} (-1)^k s_k y^{(n-k)} = q(x) \tag{10}$$

is

$$y = e^{a_n x} \int \left(\int e^{(a_{n-1}-a_n)x} \left(\int \cdots \left(\int e^{(a_1-a_2)x} \int e^{-a_1 x} q(x)dx \right) dx \cdots dx \right) dx \right) dx.$$

Proof. (By Induction). For $n = 1$, the hypothesis of the statement to be proved is that

$$y' - ay = q(x). \tag{11}$$

The equation in (11) has

$$y = e^{ax} \int e^{-ax} q(x)dx \tag{12}$$

as a general solution. Suppose the statement holds for the integer n and that

$$\sum_{k=0}^{n+1}(-1)^k s_k y^{(n+1-k)} = q(x), \tag{13}$$

where a_k for $k = 1, \ldots, n+1$ and s_k are as described above. Then by the linearity of differentiation,

$$\left(\sum_{k=0}^{n}(-1)^k s_k y^{(n-k)}\right)' - a_1\left(\sum_{k=0}^{n}(-1)^k s_k y^{(n-k)}\right) = q(x), \tag{14}$$

where $s_0 = 1$ and for each $k = 1, \ldots, n$, $s_k = \sum_\phi \prod a_{\phi(j)}$, where each ϕ is a strictly increasing function from $\{1, \ldots, k\}$ to $\{2, \ldots, n+1\}$. From the case for 1, a general solution of the equation in (14) is

$$\sum_{k=0}^{n}(-1)^k s_k y^{(n-k)} = e^{a_1 x}\int e^{-a_1 x} q(x)dx$$

and from the induction hypothesis, we obtain

$$y = e^{a_{n+1}x}\int\left[\int e^{(a_n - a_{n+1})x}\left[\int \cdots \left[\int e^{(a_1-a_2)x}\int e^{-a_1 x}q(x)dx\right]dx \cdots dx\right]dx\right]dx$$

as a general solution of (13) and the proof is complete.

The following corollaries arise naturally from the Theorem. The proofs are omitted.

Corollary. If n is a positive integer and a is a complex constant, a general solution to

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k a^k y^{(n-k)} = q(x)$$

is

$$y = e^{ax} \underbrace{\int \left(\int \cdots \left(\int}_{n \text{ integrals}} e^{-ax} q(x) dx \right) \cdots dx \right) dx.$$

Corollary. If n is a positive integer and a is a complex constant, then

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k a^k y^{(n-k)} = 0$$

has

$$y = e^{ax} \sum_{k=0}^{n-1} c_k x^{n-1-k}$$

as a general solution.

When viewed in the terminology of differential operators, the proof of the theorem is perhaps a bit more transparent. In this terminology, (10) may be written as

$$\left(\prod_{k=1}^{n} (D - a_k) \right) y = q(x)$$

and

$$\left(\prod_{k=1}^{n+1} (D - a_k) \right) y = D\left(\left[\prod_{k=2}^{n+1} (D - a_k) \right] y \right) - a_1 \left(\prod_{k=2}^{n+1} (D - a_k) \right) y,$$

from which the inductive step follows.

Reference

1. R. Euler, "A Note on a Differential Equation," *Missouri Journal of Mathematical Sciences*, 1 (1989), 26–27.

Richard E. Bayne
James E. Joseph
Myung H. Kwack
Department of Mathematics
Howard University
Washington, D.C. 20059
email: bayne@scs.howard.edu

Thomas H. Lawson
Center for Academic Reinforcement
Howard University
Washington, D.C. 20059