SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

71. [1994, 98; 1995, 91] *Proposed by Ronnie Gupton, Larry Hoehn, and Jim Ridenhour, Austin Peay State University, Clarksville, Tennessee.*

Provide a non-calculus solution to the following problem on page 530 of James Stewart's *Calculus* (2nd ed.), Brooks/Cole Publishing Company, 1991.

"A cow is tied to a silo with radius r by a rope just long enough to reach the opposite side of the silo. Find the area available for grazing by the cow."

Comment by Lamarr Widmer, Messiah College, Grantham, Pennsylvania.

It seems to me that a solution that uses a limit does not really qualify as a "non-calculus" solution. Nonetheless, it is a very interesting solution and a nice alternative to the usual solution to this cow grazing problem.

75. [1994, 160; 1995, 146] *Proposed by Leonard L. Palmer, Southeast Missouri State University, Cape Girardeau, Missouri.*

"Prove that $n(n+1)$ is never a square for $n > 0$" is a problem in *Elementary Number Theory* by Underwood Dudley. Generalize this problem by showing that $n(n+1) \neq t^k$ for t an integer and $k \geq 2$.

Comment by Kandasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

Problem 75 may be generalized in the following way.

Generalization. Suppose x and y are relatively prime and k is a positive integer such that $k > 1$ and $0 < |x-y| \leq k$. Then $xy \neq t^k$ for any integer t.

Proof. Suppose $xy = t^k$ for some integer t. Since x and y are relatively prime, $x = a^k$ and $y = b^k$ for some positive integers a and b. Then

$$x - y = a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + \cdots + b^{k-1}).$$

Note that $a, b \geq 1$ and since $|x-y| > 0$, a or b is greater than 1. Hence

$$|x-y| > |a-b|k \geq k.$$

This contradicts the fact that $|x-y| \leq k$ and the result follows.

If $x = n$, $y = n+1$ and $k \geq 2$, then $|x - y| = 1 < k$ and hence $n(n+1) \neq t^k$ for any integer t.

77. [1995, 39] *Proposed by Herta T. Freitag, Roanoke, Virginia.*

Let $P(k,n)$ denote the nth polygonal number of k "dimensions." For example, $P(3,5)$ denotes the 5th triangular number and $P(4,2)$ denotes the second square number.

(a) Find all polygonal numbers (that is, all k values) for which

$$4P(k,n)P(k,n+2)+1$$

is a square number for all n.

(b) Find all polygonal numbers for which

$$P(k,n)P(k,n+2)+1$$

is a square number for all n.

Solution by the proposer.

We will use the equation

$$P(k,n) = \frac{n}{2}[(k-2)n + (4-k)], \quad k \geq 3, /n \geq 1.$$

(a) Then,

$$4P(k,n)P(k,n+2)+1$$

$$= (k-2)^2n^4 + 2k(k-2)n^3 - (k^2 - 12k + 16)n^2 - 2k(k-4)n + 1.$$

However,

$$(an^2 + bn + c)^2 = a^2n^4 + 2abn^3 + (2ac + b^2)n^2 + 2bcn + c^2.$$

Thus, for $4P(k,n)P(k,n+2)+1$ to be a perfect square, the following relationships must hold.

$$\begin{aligned}(k-2)^2 &= a^2,\\ 2k(k-2) &= 2ab,\\ -k^2+12k-16 &= 2ac+b^2,\\ -2k(k-4) &= 2bc,\\ 1 &= c^2.\end{aligned}$$

Thus,

$$\begin{aligned}a &= k-2,\\ b &= k,\\ c &= 4-k,\\ \text{and}\ \ c &= \pm 1.\end{aligned}$$

Therefore, for $c=1$, we have $4-k=1$ and $k=3$. Hence,

$$(a,b,c)=(1,3,1),$$

whereas, for $c=-1$, $k=5$, and

$$(a,b,c)=(3,5,-1).$$

That is, for triangular numbers $P(3,n)$, we have

$$4P(3,n)P(3,n+2)+1=(n^2+3n+1)^2 \quad \text{for all} \quad n,$$

and for pentagonal numbers $P(5,n)$, it is

$$4P(5,n)P(5,n+2)+1=(3n^2+5n-1)^2 \quad \text{for all} \quad n.$$

(b) Now,

$$\begin{aligned}&P(k,n)P(k,n+2)+1\\ &\qquad=\left(\frac{k-2}{2}\right)^2 n^4 + k(k-2)/2n^3 - (k^2-12k+16)/4n^2 - k(k-4)/2n + 1,\end{aligned}$$

and, for this expression to be a square, we have

$$\begin{aligned} \left(\frac{k-2}{2}\right)^2 &= a^2, \\ k(k-2)/2 &= 2ab, \\ (-k^2+12k-16)/4 &= 2ac+b^2, \\ (-k^2+4k)/2 &= 2bc, \\ 1 &= c^2. \end{aligned}$$

Hence,

$$\begin{aligned} a &= (k-2)/2, \\ b &= k/2, \\ c &= (4-k)/2, \\ \text{and} \quad c &= \pm 1. \end{aligned}$$

If $c = 1$, $(4-k)/2 = 1$ and $k = 2$ which we'll discard ($k \geq 3$). However, if $c = -1$, $(4-k)/2 = -1$ and $k = 6$. Therefore,

$$(a, b, c) = (2, 3, -1)$$

and, for "sexagonal" numbers ($k = 6$),

$$P(6, n)P(6, n+2) + 1 = (2n^2 + 3n - 1)^2.$$

Also solved by Rob Johnson, Apple Computer, Inc.

78. [1995, 40] *Proposed by Herta T. Freitag, Roanoke, Virginia.*

Let $P(k,n)$ denote the nth polygonal number of k "dimensions." For example, $P(3,5)$ denotes the 5th triangular number and $P(4,2)$ denotes the second square number.

Let $k \geq 3$, $n \geq 1$ and let $A(k,n)$ denote the 3rd order determinant such that for $1 \leq r, s \leq 3$,

$$a_{r,s} = P(k+r-1, n+s-1).$$

Prove or disprove that for all $k \geq 3$ and $n \geq 1$, $A(k,n) = 0$.

Solution by Russell Euler, Northwest Missouri State University, Maryville, Missouri.

Since $P(k,n) = [(k-2)n^2 + (4-k)n]/2$, it is straightforward to show that $P(k+1,n) - P(k,n) = (n-1)n/2$ and $P(k+2,n) - P(k,n) = (n-1)n$. Using these results, subtract row 1 of $A(k,n)$ from rows 2 and 3 to get

$$A(k,n) = \det \begin{pmatrix} P(k,n) & P(k,n+1) & P(k,n+2) \\ (n-1)n/2 & n(n+1)/2 & (n+1)(n+2)/2 \\ (n-1)n & n(n+1) & (n+1)(n+2) \end{pmatrix}.$$

In this determinant, row 3 is a multiple of row 2. Hence, $A(k,n) = 0$ for all $k \geq 3$ and $n \geq 1$.

Also solved by Rob Johnson, Apple Computer, Inc. and the proposer.

79. [1995, 40] *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

Let a_i denote the leading digit of 2^i. Is the sequence $\{a_i\}_{i=0}^{\infty}$ eventually periodic?

Solution by Rob Johnson, Apple Computer, Inc.

Let log be the base 10 logarithm. Then if the fractional part of $i\log(2)$ is in $[\log(d), \log(d+1))$ for some integer $1 \leq d < 10$, then the leading digit of 2^i is d. Suppose the sequence of leading digits is periodic with period $n > 0$, then the fractional part of $n\log(2)$ must be 0; if not, then at some point $(i+kn)\log(2)$ will drift into a different leading digit interval, which contradicts periodicity. Thus, $m = n\log(2)$ is an integer. But then $2^n = 10^m$, which cannot be unless $m = 0$; otherwise, the right side would be divisible by 5 and the left side would not. However, this means that $n = 0$, a contradiction.

Therefore, the sequence cannot be periodic.

Also solved by the proposers.

80. [1995, 40] *Proposed by Larry Hoehn, Austin Peay State University, Clarksville, Tennessee.*

Show that

$$x^n + y^n = z^{n+1}$$

has a non-trivial integral solution for $n > 2$.

Solution I by Rob Johnson, Apple Computer, Inc. and Lawrence Somer, The Catholic University of America, Washington, D.C.

An easy but non-trivial solution for all n is $(x, y, z) = (2, 2, 2)$.

Solution II by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin and N. J. Kuenzi, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

Four trivial integral solutions for the given equation are $(x, y, z) = (0, 0, 0)$ and $(x, y, z) = (2, 2, 2)$ and $(x, y, z) = (1, 0, 1)$ and $(x, y, z) = (0, 1, 1)$.

On p. 106 of his book *Elementary Theory of Numbers* (Hafner Publishing Company, New York, 1964), Sierpinski presents the following solutions to the given equation:

$$(x, y, z) = (1 + k^n, k(1 + k^n), 1 + k^n),$$

where k is an arbitrary integer.

Solution III by Dale Woods, Reeds Spring, Missouri and the proposer.

Let

$$\begin{aligned} x &= a(a^n + b^n), \\ y &= b(a^n + b^n), \\ z &= (a^n + b^n), \end{aligned}$$

where a and b are any integers. Then

$$\begin{aligned} x^n + y^n &= [a(a^n + b^n)]^n + [b(a^n + b^n)]^n \\ &= (a^n + b^n)^n (a^n + b^n) \\ &= (a^n + b^n)^{n+1} \\ &= z^{n+1}. \end{aligned}$$

The proposer notes that the restriction to integers is not really necessary. Woods notes that this solution is quite well-known.

Solution IV by N. J. Kuenzi, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

If n is odd, we can also generate an infinite number of solutions using the following technique:

For any positive integers k and m set $w = mk^{n+1} - 1$. Since n is odd, it follows from the Binomial Theorem, that $w^n + 1$ is divisible by k^{n+1}. Set

$$\begin{aligned} x &= \frac{w^n + 1}{k^{n+1}}, \\ y &= wx, \\ z &= kx. \end{aligned}$$

Then

$$\begin{aligned} x^n + y^n &= \left[\frac{1 + w^n}{k^{n+1}}\right]^n + w^n \left[\frac{1 + w^n}{k^{n+1}}\right]^n \\ &= \left[\frac{1 + w^n}{k^{n+1}}\right]^n (1 + w^n) = \frac{(1 + w^n)^{n+1}}{(k^{n+1})^n} \\ &= \frac{(1 + w^n)^{n+1}}{(k^n)^{n+1}} = z^{n+1}. \end{aligned}$$

Example: For $n = 3$, $k = 2$ and $m = 1$, we have

$$\begin{aligned} x &= \frac{15^3 + 1}{16} = 211, \\ y &= 15x = 3165, \\ z &= 2x = 422, \text{ and} \end{aligned}$$

$$211^3 + 3165^3 = 422^4.$$

Comment I by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

In his book, *Elementary Number Theory* (second edition) (W. H. Freeman and Company, San Francisco, 1978) Underwood Dudley gives the following as problem 7 for section 17 (see pp. 193–194):

Show that there are infinitely many nontrivial solutions of

$$x^n + y^n = z^{n+1}$$

for any $n \geq 1$, namely those given by

$$x = (ac)^{rn}, \quad y = (bc)^{rn}, \quad z = c^s,$$

where

$$c = a^{rn^2} + b^{rn^2},$$

a and b are arbitrary, and r and s are chosen to satisfy

$$rn^2 + 1 = (n+1)s.$$

Does the last equation have infinitely many solutions in positive integers r, s?

(The answer to the above question is given to be: "Yes, because $(n^2, n+1) = 1$." (see p. 243 of Dudley's book.))

Comment II by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

For related information see the solutions to Problem 839 given on pp. 232–236 of the August–September 1984 issue of *Crux Mathematicorum*.