PROBLEMS

Problems, solutions, and any comments on the problems or solutions should be sent to Curtis Cooper, Department of Mathematics and Computer Science, Central Missouri State University, Warrensburg, MO 64093 or via email to ccooper@cmsuvmb.cmsu.edu.

Problems which are new or interesting old problems which are not well-known may be submitted. They may range from challenging high school math problems to problems from advanced undergraduate or graduate mathematics courses. It is hoped that a wide variety of topics and difficulty levels will encourage a number of readers to actively participate in problems and solutions. An asterisk (∗) after a number indicates a problem submitted without a solution.

Problems and solutions should be typed or neatly printed on separate sheets of paper. They should include the name of the contributor and the affiliation. Solutions to problems in this issue should be mailed no later than January 1, 1997, although solutions received after that date will also be considered until the time when a solution is published.

89. *Proposed by Stanley Rabinowitz, MathPro Press, Westford, Massachusetts.*

Let ω be a primitive 49th root of unity. Prove that

$$\prod_{\substack{k=1\\ \gcd(k,49)=1}}^{49} (1-\omega^k) = 7.$$

90. *Proposed by Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri.*

The Fibonacci polynomials, $\{U_n(x)\}$, are defined by $U_1(x) = 1$, $U_2(x) = x$, and $U_n(x) = xU_{n-1}(x) + U_{n-2}(x)$, for $n \geq 3$.

(a) Derive a Binet-like formula for $U_n(x)$.

(b) Prove that

$$(U_n(x))^2 - U_{n-1}(x)U_{n+1}(x) = (-1)^{n-1}, \quad n \geq 2.$$

(c) Find a formula for the sum

$$\sum_{k=1}^{n} (U_k(x))^2.$$

(d) Let $\{L_n\}$ be the Lucas numbers: $L_1 = 1$, $L_2 = 3$, $L_n = L_{n-1} + L_{n-2}$ $(n \geq 3)$. Prove that

$$U_n'(1) = \frac{nL_n - F_n}{5},$$

where F_n denotes the nth Fibonacci number and $U_n'(x)$ denotes the derivative of $U_n(x)$.

(e) Find a generating function for the $U_n(x)$'s, that is, a function $f(x, y)$ such that, formally,

$$f(x, y) = \sum_{n=1}^{\infty} U_n(x) y^n.$$

(f) Prove that for $n > 1$ all the zeroes of $U_n(x)$ lie along the imaginary axis.

91. *Proposed by Herta T. Freitag, Roanoke, Virginia.*

Pythagoras did not have our computational facilities for trigonometric functions (calculators or tables) at his disposal, but he may have had a "feeling" for the aesthetic beauty of the golden ratio in his soul, as he is said to have chosen the pentagram as the design for the fraternity pin of his academy.

Let R, the radius of a circle be given. How could one obtain the area of the inscribed pentagram on this basis? (Leave your answer in terms of $G = (\sqrt{5}+1)/2$, the golden ratio.)

92. *Proposed by Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri.*

It is easy to show that the homogeneous quadratic expression $A_n^2 - 2B_n^2$ is invariant for all members of the sequence $\{\gamma_n\}_{n=1}^{\infty}$, defined by $\gamma_n = (3 + 2\sqrt{2})^n = A_n + B_n\sqrt{2}$. Find a homogeneous cubic expression that is invariant for all members of the sequence $\{I_n\}_{n=1}^{\infty}$, defined by $I_n = (1 + \sqrt[3]{2} + \sqrt[3]{4})^n = A_n + B_n\sqrt[3]{2} + C_n\sqrt[3]{4}$.