ON HERMITIAN-INVERTORS

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Abstract. Specializing an invertor [20] to a linear transformation on M_n for which the image of any hermitian matrix is skew-hermitian yields a hermitian-invertor. This paper gives twelve characterizations of hermitian-invertors and lists other basic results for them. It gives a set of unifying results in a reflector setting and concludes with some remarks on $\mathbb{Z}$-linear maps in the Djoković setting [7].

1. Preliminaries. We denote the space of $m \times n$ complex matrices by $M_{m,n}$, M_n if $m = n$, with H_n and S_n representing its subsets of hermitian and skew-hermitian matrices, respectively. The sets of perhermitian and centrohermitian matrices are denoted PH_n and CH_n, respectively. A linear transformation $T\colon M_n \to M_q$ is said to be hermitian-preserving [hermitian-inverting] if and only if $T(H_n) \subseteq H_q$ [$T(H_n) \subseteq S_q$]. The sets of all such maps will be denoted $\mathcal{HP}_{n,q}$ and $\mathcal{HI}_{n,q}$, respectively, with the subscripts suppressed in context.

Theorems on linear transformations preserving some set of properties abound in the literature. For example, Volume 33, Numbers 1 and 2 of *Linear and Multilinear Algebra* [17] is a monograph dedicated to preservers. Included in this monograph is an impressively extensive list of previous preserver papers.

Hermitian-preserving linear transformations in particular have been discussed in [5], [6], [9], and [18]. These ideas permeate other matrix theory, e.g., many inertia results may be cast in terms of hermitian-preservers (cf. [2], [3], [4], [8], [9], [15], [19], [22], [23], and [24], as well as the classical results of Lyapunov [14] and Stein [21]).

The invertor concept of Siler and Hill [20] naturally specializes to the conjugate-transpose reflector $*$ on M_n, thus yielding the concept of a hermitian-invertor. As natural as this concept is, it seems to be absent from the literature (as well as the corresponding ideas of perhermitian-invertors, centrohermitian-invertors, etc.).

Hermitian-invertors fill a void in the third level of the Barker-Hill-Haertel table [1], which exists since hermitian-preservers are skew-hermitian-preservers and conversely. (A linear transformation on M_n can be uniquely represented as a sum of elements from $\mathcal{HP}$ and $\mathcal{HI}$.)

Section 2 gives twelve characterizations of hermitian-invertors in forms motivated by corresponding preserver theorems. Section 3 gives some basic invertor results, also in forms related to certain preserver theorems. Section 4 gives a unifying result in a reflector setting, characterizing invertors of H_n, PH_n, CH_n, and $M_n(\mathbb{R})$, paired with the corresponding results for preservers. Finally, Section 5 discusses results for $\mathbb{Z}$-linear matrix maps (the Djoković setting [7]).

2. Characterization Theorems. In this section we address the problem of characterizing hermitian invertors; i.e., linear transformations $T\colon M_n \to M_q$ such that $T(H)$ is skew-hermitian for all $H \in H_n$. As in [18], if $T\colon M_n \to M_q$ is linear, we let $< T > \in M_{nq}$ be the matrix representation of T with respect to the unit matrices E_{jk} (which have a one in the (j,k) position and zeros elsewhere) in M_n and M_q respectively, ordered antilexicographically.

We utilize the notation of Poluikis and Hill [18] and Oxenrider and Hill [16]. Two bijections are defined from $S = \{(j,k) \mid j = 1,\dots,q; k = 1,\dots,n\}$ to $\{1,\dots,nq\}$ by $[j,k] = (j-1)n+k$ and $< j,k >= (k-1)q+j$. These correspond to the lexicographical ordering $[(j,k) < (r,s)$ if and only if $j < r$ or $(j = r$ and $k < s)]$ and the antilexicographical ordering $[(j,k) < (r,s)$ if and only if $k < s$ or $(k = s$ and $j < r)]$, respectively, on S. We shall use $[j,k]$ and $< j,k >$ for orderings of S for different q and n; the factors q and n in their formulas will be determined by specification of the ranges of j and k.

Oxenrider and Hill [16] have studied eight element reorderings of matrices in $M_n(M_q)$ which naturally arise from rearranging the rows or columns, lexicographically or antilexicographically, into $n \times n$ blocks ordered lexicographically or antilexicographically. Four of these, which have been used to characterize hermitian-preserving [18], perhermitian-preserving [12], and centrohermitian-preserving [11] linear transformations, are defined by:

$$\Gamma(T)^{jk}_{rs} = t_{[j,k],[r,s]} = t^{jr}_{ks}, \quad \Omega(T)^{jk}_{rs} = t_{[r,s],[j,k]} = t^{rj}_{sk},$$
$$\Theta(T)^{jk}_{rs} = t_{<j,k>,<r,s>} = t^{ks}_{jr}, \quad \Psi(T)_{<r,s>,<j,k>} = t^{sk}_{rj}.$$

The following results parallel a number of characterizations of hermitian-preservers found in [5], [6], [9], and [18], of perhermitian-preservers found in [12], of centrohermitian-preservers found in [11], and κ-real and κ-hermitian preservers found in [13].

Theorem 2.1. Let $T\colon M_n \to M_q$ be a linear transformation. Then the following are equivalent:

(1) T is a hermitian-invertor.
(2) T is a skew-hermitian-invertor.
(3) There exist $A_1, \dots, A_s \in M_{q,n}$ and pure imaginary numbers $\gamma_1, \dots, \gamma_s$ such that $T(X) = \sum_{j=1}^{s} \gamma_j A_j X A_j^*$.
(4) There exist $A_1, \dots, A_s \in M_{q,n}$ and $\eta_1, \dots, \eta_s \in \{\pm i\}$ such that $T(X) = \sum_{j=1}^{s} \eta_j A_j X A_j^*$.
(5) There exist $A_1, \dots, A_s \in M_{q,n}$ and skew-hermitian $(g_{jk}) \in M_s$ such that $T(X) = \sum_{j,k=1}^{s} g_{jk} A_j X A_k^*$.
(6) $t_{rs}^{jk} = -\overline{t_{jk}^{rs}}$ for all $(j,k),(r,s) \in S$, where $< T >= ((t_{rs}^{jk}))$.
(7) The block matrix $(T(E_{jk}))_{1\le j\le n;\ 1\le k\le n}$ is skew-hermitian.
(8) $\Gamma(< T >)$ is skew-hermitian.
(9) $\Psi(< T >)$ is skew-hermitian.
(10) $\Omega(< T >)[= \Gamma(< T >^{tr})]$ is skew-hermitian.
(11) $\Theta(< T >)[= \Psi(< T >^{tr})]$ is skew-hermitian.
(12) T^* is hermitian-inverting.

Proof. Theorem 1.1 of [20] gives us (1) $\Leftrightarrow$ (2). Using a proof technique analogous to that of Theorem 1 of [9] (viz., by computing the images of T on the standard basis $\{E_{jk} + E_{kj}, i(E_{jk} - E_{kj}), E_{jj}\}$ of H_n and requiring the images to be skew-hermitian, then assuming (6) and showing these images to be skew-hermitian), we get (1) $\Leftrightarrow$ (6).

Assuming (4), a short calculation gives (1). Assuming (1), a proof analogous to that of Theorem 2 of [9] yields (4).

Upon absorbing $\sqrt{|\gamma_j|}$ in A_j, $(j = 1, \dots, s)$, (3) is a restatement of (4). Further, that (5) $\Rightarrow$ (1) and (3) $\Rightarrow$ (5) are immediate.

By Lemma 2 of [18], $\Psi(< T >) = (T(E_{jk}))$; this yields (7) $\Leftrightarrow$ (9). Since (t_{rs}^{jk}) is skew-hermitian if and only if $t_{rs}^{jk} = -\overline{t_{sr}^{kj}}$, we have (6) $\Leftrightarrow$ (8) $\Leftrightarrow$ (9).

Since a matrix is skew-hermitian if and only if its transpose is, we have (8) $\Leftrightarrow$ (10) and (9) $\Leftrightarrow$ (11).

Finally, since $\{E_{jk}\}$ is an orthonormal basis for M_n, the matrix representation for the Hilbert adjoint of T is $< T^* >=< T >^*$; this yields (1) $\Leftrightarrow$ (12).

In the above we have given summaries of the proofs of the twelve $\mathcal{HI}$ characterizations for insight into relationships among them and the corresponding $\mathcal{HP}$ theorems. We note that we could get shorter proofs by exploiting the $\mathcal{HP} = i\mathcal{HI}$ relationship.

Analogous theorems for perhermitian-invertors, centrohermitian invertors, real-invertors, κ-hermitian-invertors, and κ-real-invertors can be given.

3. Some Basic Results. The previously-mentioned "third-stage generalization" of Barker, Hill, and Haertel (Theorem 2 of [1]) gives us that every linear transformation on matrices can be uniquely expressed as the sum of a hermitian-preserver and a hermitian-invertor; viz.

Theorem 3.1. If $T \in L(M_n, M_q)$, then there exist unique linear functions $H \in \mathcal{HP}_{n,q}$ and $K \in \mathcal{HI}_{n,q}$ such that $T = H + K$.

This result turns out to be a specialization of Theorem 2.2 of [20]. Analogous results include Theorem 2.11 of [13], Theorem 2.11 of [11], Theorem 2.11 of [12], the Toeplitz (cartesian) decomposition [15], and the Hadamard decomposition $A = R_1 + iR_2$ with $R_1, R_2 \in M_n(\mathbb{R})$.

Further, since a linear transformation is hermitian-preserving if and only if it is skew-hermitian-preserving, this theorem fills a void in an extension of the Barker-Hill-Haertel table in the third-stage generalization of {pure imaginary number} $\rightarrow$ {skew-hermitian matrices} $\rightarrow$ (?).

Results from Hill [9] and Djoković [7] motivate the following three results. Their proofs are similar to the corresponding results for hermitian-preservers in [9].

Theorem 3.2. If $T \in \mathcal{HI}$, then $T(X^*) = -(T(X))^*$ for all $X \in M_n$.

Theorem 3.3. If λ is an eigenvalue of algebraic [geometric] multiplicity r of $T \in \mathcal{HI}$, then $-\overline{\lambda}$ is also an eigenvalue of algebraic [geometric] multiplicity r of T.

Theorem 3.4. If $T \in \mathcal{HI}_n$, then the odd elementary symmetric functions $E_1, E_3, \ldots$ of (the eigenvalues of) T are pure imaginary, whereas the evens $E_2, E_4, \ldots$ are real. In particular, $\mathrm{tr}(T)$ is pure imaginary and $\det(T)$ is pure imaginary [real] if n is odd [even].

Analogous to the hermitian-preserving case [Theorem 4 of 18], $\sum_{j,k=1}^{s} g_{jk} A_j \otimes \overline{A_k}$ can represent a hermitian-invertor without (g_{jk}) being skew-hermitian. Again linear independence of $A_1, \ldots, A_s$ is sufficient. The results can be stated formally as follows:

Theorem 3.5. Let $A_1, \ldots, A_s$ be linearly independent in $M_{q,n}$. Then

(1) $\sum_{j,k=1}^{s} g_{jk} A_j \otimes \overline{A_k}$ represents a hermitian-invertor if and only if (g_{jk}) is skew-hermitian.

(2) $\sum_{j=1}^{s} \gamma_j A_j \otimes \overline{A_j}$ represents a hermitian-invertor if and only if $\gamma_1, \ldots, \gamma_s$ are pure imaginary.

4. Characterizations in a Reflector Setting. As in [20], a reflector is an additive, involutory map. Any reflector σ on a vector space over a field of characteristic not equal to 2 induces the two additive, idempotent maps $\text{spec}_\sigma(x) := \frac{1}{2}(x + \sigma(x))$ and $\text{skew}_\sigma(x) := \frac{1}{2}(x - \sigma(x))$. The ranges of spec_σ and skew_σ consist of the elements fixed or negated, respectively, by σ.

Let σ be a conjugate-homogeneous reflector on M_n and let τ be defined on $L(M_n)$ by $(\tau(T))(A) := \sigma(T(\sigma(A)))$. Then τ is a conjugate-homogeneous reflector on $L(M_n)$. By Theorem 3.1 of [20], $\text{spec}_\tau L(M_n)$ and $\text{skew}_\tau L(M_n)$ consist of the linear preservers and invertors, respectively, of $\text{spec}_\sigma M_n$ (equivalently, of $\text{skew}_\sigma M_n$). Further, by [20, p. 3], $\text{skew}_\tau L(M_n) = i\ \text{spec}_\tau L(M_n)$.

These observations, together with the facts that the reflectors $\sigma(A) := \overline{A}$, A^*, JA^*J, and $J\overline{A}J$ are conjugate-homogeneous, where $J = (\delta_{j,n-k+1})$, give us the following result:

Theorem 4.1. A linear transformation T on M_n *inverts*

(1) H_n (and skew-hermitian matrices)

(2) PH_n (and skew-perhermitian matrices)

(3) CH_n (and skew-centrohermitian matrices)

(4) $M_n(\mathbb{R})$ (and $M_n(i\mathbb{R})$),

respectively, if and only if there exists $U \in L(M_n)$ such that

(1) $T(X) = U(X) - (U(X^*))^*$

(2) $T(X) = U(X) - J(U(JX^*J))^*J$

(3) $T(X) = U(X) - J\overline{U(J\overline{X}J)}J$

(4) $T(X) = U(X) - \overline{U(\overline{X})}$.

We now have a first in that this "invertor theorem" motivates a corresponding "preserver theorem."

Theorem 4.2. A linear transformation T on M_n *preserves*

(1) H_n (and skew-hermitian matrices)
(2) PH_n (and skew-perhermitian matrices)
(3) CH_n (and skew-centrohermitian matrices)
(4) $M_n(\mathbb{R})$ (and $M_n(i\mathbb{R})$),
respectively, if and only if there exists $U \in L(M_n)$ such that
(1) $T(X) = U(X) + (U(X^*))^*$
(2) $T(X) = U(X) + J(U(JX^*J))^*J$
(3) $T(X) = U(X) + J\overline{U(J\overline{X}J)}J$
(4) $T(X) = U(X) + \overline{U(\overline{X})}$.

5. Characterizations in the Djoković Setting.

The setting of this section is as in Djoković [7]. Let $\mathbb{Z}$ be the center of a division ring D. Assume that D has finite dimension over $\mathbb{Z}$ and that char $D \neq 2$. Let $\mathcal{J}$ denote an additive, involutory, reverse-multiplicative map on D.

Define $*$ on $M_{n,q} = M_{n,q}(D)$ by $A^* := (b_{jk})$, where $b_{jk} = a_{kj}^{\mathcal{J}}$. Then $*$ gives an additive, involutory, reverse-multiplicative map on $M_{n,q}$.

A matrix $A \in M_n$ is said to be *hermitian* [*skew-hermitian*] if and only if $A = A^*$ [$A = -A^*$]; a $\mathbb{Z}$-linear map $T\colon M_n \to M_q$ is said to be *hermitian-preserving* [*hermitian-inverting*] if and only if $T(A)$ is hermitian [skew-hermitian] whenever A is hermitian.

Djoković's Theorem (ii) may be restated by renaming his $A_1, \ldots, A_p$, $B_1, \ldots, B_p$ as $A_1, \ldots, A_{2p}$ as follows:

Theorem 5.1. A $\mathbb{Z}$-linear map $T\colon M_n \to M_q$ is hermitian-inverting if and only if there exist $A_1, \ldots, A_{2p} \in M_{n,q}$ such that $T(X) = \sum_{j,k=1}^{2p} g_{jk} A_j^* X A_k$, where $(g_{jk}) =$ perdiag$\{1, 1, \ldots, 1, -1, -1, \ldots, -1\}$, where 1 and -1 each appear with multiplicity p.

Since this (g_{jk}) is skew-hermitian and since its eigenvalues are $\pm i$ (each with multiplicity p), we have characterizations in this setting analogous to (3), (4), and (5) of Theorem 2.1.

Further, since this (g_{jk}) is perhermitian we also have characterizations analogous to (3), (4), and (5) of Theorem 4.1 of [12].

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