ON THE DECOMPOSITION OF THE UNIT INTERVAL

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In a recent article [1] in this journal, J. Bennish raises a question of decomposing the unit intervals into a large number of disjoint sets in such a way that the intersection of each set with every open interval is, in some sense, large. To this end, he constructs a decomposition of the unit interval into countably many disjoint subsets such that the intersection of every one of these subsets with any open interval has a positive Lebesgue measure. In this note, we construct a decomposition of the unit interval into uncountably many disjoint subsets, such that every one of these subsets has the Lebesgue measure 0, but the intersection of every one of these subsets with any open interval has positive Hausdorff dimension.

Every $x \in I$ can be written in binary expansion:

$$x = \sum_{j=1}^{\infty} \epsilon_j(x) 2^{-j}, \ \ \epsilon_j(x) = 0 \text{ or } 1,$$

and the expansion is unique if we stipulate that infinitely many ϵ_j's must be 0. For each $0 \leq p \leq 1$, let U_p be the set of those x's for which the proportion of 1's in this expansion is p. More precisely, $x \in U_p$ if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \epsilon_j(x) = p. \tag{1}$$

Let λ denote the Lebesgue measure. It is a well known fact that $\lambda(U_{1/2}) = 1$ and $\lambda(U_p) = 0$ for $p \neq 1/2$. It is also known that the Hausdorff dimension of U_p is $-(p \log_2 p + q \log_2 q) > 0$, if $0 < p < 1$, where $p + q = 1$ (all logs are in base 2). See [2] for all of this. Our basic decomposition of the unit interval is the family of set $\mathcal{U} = \{U_p \mid p \neq 1/2, \ 0, \ \text{or } 1\}$. The family $\mathcal{U}$ does not exhaust all of I. Of course, $U_{1/2}$, U_0, and U_1, are disjoint from all the sets of $\mathcal{U}$. Equally well, all the points x for which the limit (1) does not exist altogether are not in any of the sets of $\mathcal{U}$. There is, however, only a continuum number of x's not in $\mathcal{U}$ and there is a continuum number of sets in $\mathcal{U}$. We take all the points not in $\mathcal{U}$ and distribute them among the sets in $\mathcal{U}$, one point per set. Adding a single point to a set does not change its measure nor the Hausdorff dimension. For simplicity, we also denote these new sets by U_p. It is clear that each set U_p is of measure 0, and it is easy to show that for any interval J, the Hausdorff dimension of $J \cap U_p = -(p \log_2 p + q \log_2 q) > 0$. Indeed, let p be fixed and put $\gamma = -(p \log_2 p + q \log_2 q)$. For any integer $k > 0$, let $t_{k,m} = m2^{-k}$, $m = 0, 1, 2, 3, \dots, 2^k - 1$. Let

$$I_{k,m} = U_p \cap [t_{k,m}, t_{k,m+1}).$$

Then $I_{k,m} = I_{k,0} + t_{k,m}$, since for any $t \in [t_{k,0}, t_{k,1})$, t and $t + t_{k,m}$ have binary expansions which differ only in a finite number of places. Thus, either both of them are in U_p, or neither of them are in U_p. Hence, all the sets $I_{k,m}$ have the same Hausdorff dimension, because the Hausdorff dimension is clearly invariant under translation. It is also clear that the Hausdorff dimension of a finite union of disjoint sets is the maximum of the dimensions of these sets. Since

$$U_p = \bigcup_m I_{k,m},$$

the Hausdorff dimension of each $I_{k,m}$ is γ. If J is any open interval, $U_p \cap J$ contains the set $I_{k,m}$ for some k, m. Thus, dimension of $U_p \cap J$ is at least γ, but it is also at most γ, because $U_p \cap J$ is contained in U_p.

It would be interesting to construct a decomposition of the unit interval into an uncountable number of sets such that the intersection of each such set with any open interval has Hausdorff dimension 1.

References

1. J. Bennish, "Decomposition of the Line into Countably-Many Measure Theoretic Dense Sets," *Missouri Journal of Mathematical Sciences*, 5 (1993), 123–125.

2. P. Billingsley, *Ergodic Theory and Information*, Wiley, New York, NY, 1965.

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