ON WEAK* SEPARABLE SUBSETS OF DUAL BANACH SPACES

Joe Howard

New Mexico Highlands University

Let X denote a Banach space, B_X its closed unit ball, and X' be the Banach dual space of X. sX is the set of all weak* sequentially continuous functionals on X', and tX the set of all $f \in X''$ which are weak* continuous on all bounded weak*-separable subsets of X'. The sets sX and tX are closed subspaces of X''. The relation $JX \subseteq tX \subseteq sX \subseteq X''$ holds where J is the canonical map of X into X''. It is well known that $JX = sX$ is equivalent to X' being a Mazur space (weak* sequentially continuous functionals on X' are weak* continuous); that $sX = X''$ is equivalent to X being a Grothendieck space (in X', weak* convergent sequences are weakly convergent). By considering the set tX, we have the following.

Theorem 1. The following are equivalent.

(a) $tX = X''$.

(b) Every operator $T: X \to l_\infty$ is weakly compact.

(c) X is a reflexive Banach space.

Proof. (a) $\Rightarrow$ (b) Let $f \in X''$. Then f is continuous on all bounded separable subsets A' of X' with the weak* topology. Let $T: X \to l_\infty$ be an operator. Since B'_{l_∞} is weak* separable, $A' = T'(B'_{l_\infty})$ is a bounded weak* separable set. Since f is weak* continuous on A' by hypothesis, $T''(f) = f \circ T'$ is weak* continuous on B'_{l_∞}; thus, $T''(f) \in l_\infty$. Then $T''X'' \subseteq Jl_\infty$ and T is weakly compact.

(b) $\Rightarrow$ (c) A bounded sequence (x'_n) in X' defines an operator

$$T: X \to l_\infty \text{ by } Tx = (< x'_n, x >).$$

Since every operator is weakly compact, we obtain that every bounded sequence in X' contains a weakly convergent subsequence. By Eberlein's Theorem, this implies X', hence, X is reflexive.

(c) $\Rightarrow$ (a) is clear.

From this theorem, it follows that a Grothendieck space with the property that $tX = sX$ must be reflexive. We now consider this property.

A set $A' \subseteq X'$ is weak*-M compact if for every bounded sequence (x'_n) in A', the weak* closure of (x'_n) has a weak* convergent subsequence. Note that if a set is weak* sequentially compact, then the set is weak*-M compact, but not conversely.

Theorem 2. The following are equivalent.

(a) $tX = sX$.

(b) Every operator $T: X \to l_\infty$ is such that $T': l'_\infty \to X'$ maps bounded sets of l'_∞ into weak*-M compact subsets of X'.

(c) $B_{X'}$ is weak*-M compact.

The proof of this theorem is very similar to the previous theorem. A limited set A of X is a set such that for every weak*-null sequence (x'_n) in X, we have $x'_n(x) \to 0$ uniformly for $x \in A$. If all limited subsets of E are relatively norm compact, then X is said to have the Gelfand-Phillips (GP) property. By using Theorem 2 and Corollary 2.3 of [1], it follows that if $tX = sX$, then X has the GP property.

Corollary 3. If X' is a Mazur space, then X has the GP property.

The converse of Corollary 3 is not true as the space $C[0,\omega_1]$ has the GP property ($B_{C[0,\omega_1]'}$ is weak* sequentially compact), but $C[0,\omega_1]'$ is not a Mazur space (see [2]).

X is called realcompact if the weak topology on X is realcompact, that is, homeomophic to a closed subset of the product $\mathbb{R}^I$ for some set I.

Theorem 4. Suppose X is realcompact and $tX = sX$. Then X' is a Mazur space.

Proof. By [3] (see Proposition 4.3 of [2]), we know that X is realcompact if and only if JX coincides with the space rX of all $f \in X''$ such that the restriction of f to every weak* separable closed subspace of X' is continuous. Since each weak* separable closed subspace of X' can be generated by the weak* closure of a bounded sequence (x'_n), then in each weak* separable subspace is a weak* convergent subsequence (since $tX = sX$), and conversely. Hence, $JX = sX$, so X' is a Mazur space.

References

1. L. Drewnowski, "On Banach Spaces with the Gelfand-Phillips Property," *Math. Z.*, 193 (1986), 405–411.

2. G. A. Edgar, "Measurability in a Banach Space, II," *Indiana University Mathematics Journal*, 28 (1979), 559–579.

3. M. Valdivia, *Topics in Locally Convex Space*, North Holland.