SEQUENTIAL G_δ-SETS AND MEASURES

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For any set Y we equip the power set $P(Y)$ with a compact Hausdorff topology by taking as subbasic open sets all sets of the forms $\{D \subseteq Y \mid y \in D\}$ and $\{D \subseteq Y \mid y \notin D\}$ where y varies throughout Y. For any class X of sets, the union of this class is a set $\cup X$ and its power set $P(\cup X)$ may be equipped with a topology as above. Considered as a subset of $P(\cup X)$, X might be both sequentially closed in $P(\cup X)$ and a sequential G_δ-set (the intersection of countably many sequentially open sets) in $P(\cup X)$. If this is the case, we say that X is a *Mazur class.*

For any set Y, we say that Y is *Mazur reducible* provided every Mazur class of subsets of Y that contains all finite subsets of Y must also contain Y itself. A cardinal number m is said to be Mazur reducible if there is a Mazur reducible set of cardinality m. (If a set is Mazur reducible, then so are all sets of the same cardinality as this set.)

Mazur [2] proved that all cardinal numbers less than the smallest inaccessible cardinal number are Mazur reducible. Ulam [3] proved that each nonmeasurable cardinal number is non-realmeasurable provided 2^ω is non-realmeasurable.

For any set Y, we say that Y is *modified Mazur reducible* provided every Mazur class of subsets of Y that is finitely additive and contains all finite subsets of Y must also contain Y itself. As above, we say that a cardinal number m is modified Mazur reducible if there is a modified Mazur reducible set of cardinality m.

It follows from [1, Theorem 1] that all Mazur reducible cardinals are modified Mazur reducible.

<u>Theorem</u>. Every nonmeasurable cardinal number is modified Mazur reducible if and only if 2^ω is modified Mazur reducible.

<u>Proof</u>. Since 2^ω is nonmeasurable, it remains to prove that if 2^ω is modified Mazur reducible, then m is modified Mazur reducible for every nonmeasurable cardinal number m.

<u>Lemma 1</u>. Let Y denote a set of nonmeasurable cardinality. If X is a Mazur class of subsets of Y that is finitely additive and contains all finite subsets of Y, then for every

$B \in P(Y) - X$ there is a partition of B into two (disjoint) sets both belonging to $P(Y) - X$ or both belonging to X.

Proof. Suppose the conclusion is false; i.e., there is a set $B \in P(Y) - X$ such that for every partition S/T of B exactly one of S and T belongs to X. But this defines a measure on B and hence, is inconsistent with the fact that Y has nonmeasurable cardinality.

Lemma 2. Suppose Y is any set, and X is a Mazur class of subsets of Y that is finitely additive and contains all finite subsets of Y, and for every $B \in P(Y) - X$ there is a partition of B into two sets both belonging to $P(Y) - X$ or both belonging to X. There there is a family $\{A_\xi \mid \xi \in \Xi\}$ of mutually disjoint members of X such that card $(\Xi) \leq 2^\omega$ and $Y = \cup\{A_\xi \mid \xi \in \Xi\}$.

Proof. Suppose the conclusion is false; i.e., every representation of Y as a union of mutually disjoint members of X must be formed by a family of cardinality greater than 2^ω. By hypothesis, the set Y partitions into $B_1^0, B_2^0 \in P(Y) - X$. Take $S_0 = \emptyset$. Again B_1^0 and B_2^0 each partition into sets $B_i' \in P(Y) - X$ $(1 \leq i \leq 4)$ and we take $S_1 = \emptyset$ again.

Assume that $S_\xi \subseteq X$ and $B_n^\xi \in P(Y) - X$ have been defined for every $\xi < \eta < \omega_1$ and satisfy the following conditions:

(1) card $S_\xi \leq 2^\omega$, and the sets in S_ξ are mutually disjoint;

(2) $S_{\xi'} \subseteq S_\xi$ for $\xi' < \xi$;

(3) $B_n^\xi \cap B_m^{\xi'} = \emptyset$, or $B_n^\xi \subseteq B_m^{\xi'}$ and $B_m^{\xi'} - B_n^\xi \in P(Y) - X$ for $\xi' < \xi$; and

(4) $(\cup S_\xi) \cup (\cup_n B_n^\xi) = Y$, $(\cup S_\xi) \cap (\cup_n B_n^\xi) = \emptyset$.

Consider $C = \{H \in P(Y) - \{\emptyset\} \mid$ there are n_ξ with $H = \cap\{B_{n_\xi}^\xi \mid \xi < \eta\}\}$. By assumption and condition (1), $Y \neq \cup S_\xi$ for every $\xi < \eta$. Let $p \in Y - \cup S_\xi$ for every $\xi < \eta$. Thus, for every $\xi < \eta$ there is a set $B_{n_\xi}^\xi$ containing p. Thus, $C \neq \emptyset$. The sets in C are mutually disjoint since their defining sequences must differ for some $\xi < \eta$.

Let $S_\eta = (\cup\{S_\xi \mid \xi < \eta\} \cup \{H \in C \mid H \in X\}$. Then $Y = (\cup S_\eta) \cup (\cup(C - S_\eta))$ and $C - S_\eta$ is a countable family of sets in $P(Y) - X$ because $P(Y) - X$ is a countable union of sequentially closed sets, X contains all finite subsets of Y, the sets in C are mutually disjoint, and the pigeon-hole principle holds. Since card $(S_\eta) \leq 2^\omega$, $C - S_\eta \neq \emptyset$. Applying the hypothesis to each set in $C - S_\eta$ yields a countable sequence $B_1^\eta, B_2^\eta, \cdots$. It is routine to verify that conditions (1)–(4) are satisfied.

Since $Y \neq \cup\{\cup\{S_\xi \mid \xi < \omega_1\}\}$, there is a $p \in Y - \cup\{\cup\{S_\xi \mid \xi < \omega_1\}\}$. Thus, there is an n_ξ for every $\xi < \omega_1$, for which $p \in \cap\{B_{n_\xi}^\xi \mid \xi < \omega_1\}$. Then $\{B_{n_\xi}^\xi - B_{n_{\xi+1}}^{\xi+1} \mid \xi < \omega_1\}$ is an uncountable, mutually disjoint collection of elements of $P(Y) - X$.

Because $P(Y) - X$ is a countable union of sequentially closed sets, one of these sets, say X_0, must therefore contain uncountably many elements of the above mutually disjoint collection, and any sequence, with distinct terms, of these sets converges to the empty set. This is a contradiction since the empty set belongs to X, X_0 is sequentially closed, and X is disjoint from X_0.

To complete the proof of the theorem, we suppose Y is a set of nonmeasurable cardinality and that X is a Mazur class of subsets of Y that is finitely additive and contains all finite subsets of Y. By Lemmas 1 and 2, there is a family $\{A_\xi \mid \xi \in \Xi\}$ of mutually disjoint members of X such that $Y = \cup\{A_\xi \mid \xi \in \Xi\}$ and card $(\Xi) \leq 2^\omega$.

Define a map ϕ from Y onto Ξ by $\phi(y) = \xi$ for $y \in A_\xi$. It is easy to check that $X_\phi = \{\Gamma \subseteq \Xi \mid \phi^{-1}[\Gamma] \in X\}$ is a Mazur class of subsets of Ξ that is finitely additive and contains all finite subsets of Ξ.

By hypothesis, 2^ω is modified Mazur reducible (and hence, all cardinal numbers less than 2^ω are also) so that $\Xi \in X_\phi$. Thus, $Y = \phi^{-1}[\Xi] \in X$, which was to be proved.

References

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