ULTRAFILTERS AND TOPOLOGICAL ENTROPY OF COMPLEMENTARY TOPOLOGIES

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Abstract. A topology τ on a set X is called a complementary topology if for each open set U in τ, its complement $X - U$ is also in τ. These topologies and maximal ideals were characterized by this author. In this paper the relations between maximal ideals of τ as a Boolean ring and ultrafilters in τ as a complementary topology have been investigated. Finally these relations have been characterized.

1. Introduction. Let τ be a topology on a set X. Then τ is called a complementary topology (comp-topology) if for each $U \in \tau$, its complement $X - U$ is also in τ. In [3], these spaces along with maximal ideals have been characterized. In this article relations between maximal ideals and ultrafilters are investigated and characterized.

2. Properties of Complementary Topology. To characterize the comp-topology, we state some lemmas which the first lemma has been proved in [3].

<u>Lemma 1</u>. In a comp-topology, the intersection of an arbitrary collection of open sets is an open set.

Note that the converse of the above lemma is not true. For example if $X = \mathbb{R}$, then the family $\{(-n, n) \mid n \in \mathbb{N}\}$ is a basis for a topology on $\mathbb{R}$ which is not a comp-topology. But an arbitrary intersection of open sets is open in this topology.

<u>Lemma 2</u>. If τ is a non-trivial comp-topology on a set X, then τ admits a unique basis which forms a partition for the space X. This partition is called "*disjoint basis.*"

<u>Proof</u>. Assume τ is a comp-topology. Let $\beta = \{U_\alpha \mid \alpha \in A\}$ be a collection of all mutually pairwise disjoint non-empty open sets in X. A question may arise about the method of selecting these open sets. For any $x \in X$, let $\{V_\gamma \mid \gamma \in B\}$ be a collection of all open sets containing x. Then we select their intersection U_α which is open by Lemma 1. This open set is a member of β. If we consider the collection $\{V_\gamma - U_\alpha \mid \gamma \in B\}$ which is also well ordered by inclusion. This subcollection has the smallest element which is a member of β. This process will give us a family β of mutually pairwise disjoint open sets.

The collection $\beta = \{U_\alpha \mid \alpha \in A\}$ forms a partition for X. Assume $X - \cup_{\alpha \in A} U_\alpha \neq \emptyset$, then $V = X - \cup_{\alpha \in A} U_\alpha$ is an open set in comp-topology τ. Since $V \cap U_\alpha = \emptyset$ for each $\alpha \in A$, we conclude that $V \in \beta$, a contradiction. So $X = \cup_{\alpha \in A} U_\alpha$. To show β is a basis for τ, let U be an open set in X and $x \in U$, then there exists a unique $U_\alpha \in \beta$ such that $x \in U_\alpha$. The open set $U \cap U_\alpha \neq \emptyset$ and is contained in U_α. By minimality of U_α, we must have $U \cap U_\alpha = U_\alpha$ which implies $U_\alpha \subset U$. The uniqueness of this basis follows from the fact that for any open set U and any element of disjoint basis U_α if $U_\alpha \cap U \neq \emptyset$ then $U_\alpha \subset U$.

Lemma 3. If the topology τ on a set X admits a basis which forms a partition for X, then τ is a comp-topology.

Proof. Assume $\beta = \{U_\alpha \mid \alpha \in A\}$ forms a partition for X and is a basis for τ. Let U be an arbitrary open set in X. Then $U = \cup_{\alpha \in B \subseteq A} U_\alpha$ and $X = \cup_{\alpha \in A} U_\alpha$ and $X - U = \cup_{\alpha \in A} U_\alpha - \cup_{\alpha \in B \subset A} U_\alpha = \cup_{\alpha \in A-B} U_\alpha$ which implies that $X - U$ is open in X.

Lemma 2 and Lemma 3 can be employed to prove the following theorem which characterize the comp-topological space.

Theorem 1. Let τ be a non-trivial topology on a set X, then τ is a comp-topology if and only if τ admits a unique basis which forms a partition for the set X.

Throughout this article this unique basis is called disjoint basis induced by the comp-topology τ on a set X.

Every subspace of a comp-topological space is a comp-topological space and comp-topology has topological property, i.e. the homeomorphic image of any comp-topology is a comp-topology. Also if τ_X and τ_Y are comp-topologies on X and Y, respectively then $\tau_{X \times Y}$ is a comp-topology. Indeed if $\{U_\alpha \mid \alpha \in A\}$ and $\{V_\beta \mid \beta \in B\}$ are disjoint basis induced by τ_X and τ_Y, respectively then the family $\mathcal{U} = \{U_\alpha \times U_\beta \mid (\alpha, \beta) \in A \times B\}$ is a disjoint basis for the topology $\tau_{X \times Y}$. (see [4], p. 87).

The above statement is not true for the Cartesian product topology. Indeed if $\{X_\gamma\}_{\gamma \in \Lambda}$ is an indexed family of comp-topological space and cardinality (Card A) of λ is greater than or equal to $\aleph_0$, then $\prod_{\lambda \in \Lambda} X_\lambda$ is not a comp-topological space. For example, let $\lambda = \mathbb{N}$, the set of positive integers and for each $n \in \mathbb{N}$, let $X_n = \{0, 1\}$ with the discrete topology. The Cartesian topology on $\prod_{n=1}^{\infty} X_n$ is neither discrete nor comp-topology.

Disconnectedness of a non-trivial comp-topology implies that these topologies do not have the fixed point property.

It is known that if a space admits a basis with finitely many elements then X is compact. In the case of a comp-topology, this can be stated as follows:

Proposition 1. If X is a comp-topology with the disjoint basis $\{U_\alpha \mid \alpha \in A\}$ then X is compact if and only if the cardinality of index set A (Card A) is finite.

Proposition 2. A non-trivial comp-topology is T_1 if and only if it is discrete.

Proposition 3. A comp-topology is T_2 if and only if it is T_1.

Proposition 4. A comp-topology is Tychonoff (regular and T_1) if and only if it is discrete.

Let R be a relation on X defined by $(x, y) \in R$ if there exists a unique U_α such that $x, y \in U_\alpha$. It is clear that R is an equivalence relation on X. It is straight forward to see that $\frac{X}{R}$, the set of equivalence classes of R under the natural map $\phi\colon X \to \frac{X}{R}$, is a discrete space and so is a k space.

The following theorem is useful in computing the topological entropy of homeomorphism with respect to an open covering.

Theorem 2. Let τ be a comp-topology on a set X with $\{U_\alpha \mid \alpha \in A\}$ as the disjoint basis. Then a *bijection* function $h\colon X \to X$ is a homeomorphism if and only if for any element U_α of disjoint basis there are U_β and U_γ in this basis such that $U_\alpha = h^{-1}(U_\beta)$ and $U_\alpha = h(U_\gamma)$.

Proof. Assume h is a homeomorphism. Let U_α be an arbitrary element of the disjoint basis. There is an element U_β in this basis such that $U_\beta \subseteq h(U_\alpha)$ since $h(U_\alpha)$ is open in X. $h^{-1}(U_\beta) \subseteq h^{-1}(h(U_\alpha)) = U_\alpha$. Since U_α is the smallest open set in X and $h^{-1}(U_\beta)$ is open, this implies that $h^{-1}(U_\beta) = U_\alpha$. $h^{-1}(U_\alpha)$ is also open in X because h is continuous so there is U_γ in the disjoint basis such that $U_\gamma \subseteq h^{-1}(U_\alpha)$, which implies $h(U_\gamma) \subseteq h(h^{-1}(U_\alpha)) = U_\alpha$. Again by minimality of U_α we end up that $h(U_\gamma) = U_\alpha$.

To show h is a homeomorphism, it suffices to show that h is surjective, open, and continuous. Let $y \in X$. Then there exists a unique U_α in the disjoint basis such that $y \in U_\alpha$. By assumption $U_\alpha = h(U_\gamma)$, which shows that h is surjective. For continuity, let U be an open set in X and $x \in h^{-1}(U)$, then $h(x) \in U_\alpha \subseteq U$. Thus, by assumption $h(x) \in U_\alpha = h(U_\gamma) \subseteq U$ for some U_γ in the disjoint basis. But the later relation implies that $x \in U_\gamma \subseteq h^{-1}(U)$ and $h^{-1}(U)$ is open in X. To show h is open let U be an open set in X and $y = h(x) \in h(U)$, then there is U_α in the disjoint basis such that $x \in U_\alpha \subseteq U$. By assumption there is U_β such that $x \in U_\alpha = h^{-1}(U_\beta) \subseteq U$. This implies that $h(x) \in U_\beta \subseteq h(U)$ and therefore h is an open map.

2. Comp-topology as Boolean Rings, Ideals, and Filters. Let τ be a comp-topology with the disjoint basis $\{U_\alpha \mid \alpha \in A\}$. In [3], it has been shown that τ with the operations $+, \cdot$ defined by $A + B = (A - B) \cup (B - A)$ and $A \cdot B = A \cap B$ for any $A, B \in \tau$ is a Boolean ring. Here we show the relation between ideals (dual ideals) of τ as a ring and the filters in τ. Let us recall that a non-empty family $\mathcal{F}$ of non-empty subsets of X is called a filter if i) $A \subset B$ and $A \in \mathcal{F}$ then $B \in \mathcal{F}$; ii) For any, $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$. A filter is said to be an ultrafilter or a maximal filter if there is no strictly finer filter $\mathcal{G}$ than $\mathcal{F}$. Let (X, τ) be a topological space and let $\mathcal{F}$ be a collection of subsets of X. $\mathcal{F}$ is called a filter in τ if i) $A \in \mathcal{F}$, $B \in \tau$, and $A \subset B$, then $B \in \mathcal{F}$; ii) For any $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$.

Also recall that $I \subset \tau$ is an ideal if $(I, +)$ forms an abelian group and for each $A \in I$ and each $B \in \tau$, $A \cdot B \in I$. An ideal is said to be a maximal ideal if $I \neq \tau$ and I is not contained in any other ideal of τ.

Theorem 3. Let I be a proper subset of the Boolean ring $(\tau, +, \cdot)$ where τ is a comp-topology on X. Then I is an ideal if and only if $\mathcal{F} = \{X - K \mid K \in I\}$ is a filter in τ.

Proof. Assuming I is an ideal, we show that $\mathcal{F}$ as a subset of τ is a filter in τ. It is clear that $\emptyset \notin \mathcal{F}$. Let $B \in \tau$, $A \subset B$ and $A \in \mathcal{F}$. Then $A = X - K$ for some $K \in I$. It suffices to show that there is a $K^* \in I$ such that $B = X - K^*$. Take $K^* = K - B$. Then

$$X - K^* = X - (K - B) = X - (K \cap (X - B)) = (X - K) \cup B = A \cup B = B.$$

Since $K^* = K - B = K \cap (X - B)$ and $X - B \in \tau$ and I is an ideal, then

$$K \cdot (X - B) = K \cap (X - B) = K^* \in I.$$

Now let $A, B \in \mathcal{F}$. Then by the definition of $\mathcal{F}$, there exist K_1 and K_2 in I such that $A = X - K_1$, $B = X - K_2$. $A \cap B = (X - K_1) \cap (X - K_2) = X - (K_1 \cup K_2)$ by DeMorgan Law. If we show $K_1 \cup K_2 \in I$, then the if part of the theorem has been proved. To show this, it is clear that $K_1 + K_2 = (K_1 - K_2) \cap (K_2 - K_1)$ and $K_1 \cdot K_2 = K_1 \cap K_2$ are in I, and consequently $(K_1 + K_2) + K_1 \cdot K_2 = K_1 \cup K_2 \in I$.

We now show that if $\mathcal{F}$ is a filter, then I as a proper subset of τ is an ideal. Let $K_1, K_2 \in I$. Then $A = X - K_1$, $B = X - K_2$ are in the filter $\mathcal{F}$ and so $A \cap B = X - (K_1 \cup K_2)$ is also in $\mathcal{F}$. This shows that $K_1 \cup K_2 \in I$. It is clear that $(K_1 - K_2) \cup (K_2 - K_1) \subset K_1 \cup K_2$ and $X - (K_1 \cup K_2) \subset X - ((K_1 - K_2) \cup (K_2 - K_1))$. Since $\mathcal{F}$ is a filter and $X - (K_1 \cup K_2) \in \mathcal{F}$. Then $X - ((K_1 - K_2) \cup (K_2 - K_1)) \in \mathcal{F}$ which shows $K_1 + K_2 \in I$. So I is closed with respect to $+$. It is straight forward to show I is an abelian group with respect to the addition. Let A be an arbitrary element in τ and $K \in I$, by observing that $X - K \subset (A \cap K)$ and $X - K$ is in $\mathcal{F}$ where $\mathcal{F}$ is a filter, we conclude that $A \cap K = A \cdot K \in I$ and the proof is completed.

In [3], it has been shown that if we consider the comp-topology with the disjoint basis $\{U_\alpha \mid \alpha \in A\}$ as a Boolean ring, then for any fixed $\alpha_0 \in A$, the set $I_{\alpha_0} = \{U \mid U \cap U_{\alpha_0} = \emptyset\}$ is a maximal ideal in τ. In the next theorem we will characterize some of the ultrafilters in this Boolean ring.

Theorem 4. Let τ be a comp-topology with $\{U_\alpha \mid \alpha \in A\}$ as the disjoint basis, and let α_0 be a fixed element in A. Then the set

$$\mathcal{F} = \{X - U \mid U \in \tau \text{ and } U \cap U_{\alpha_0} = \emptyset\}$$

is an ultrafilter for X.

Proof. Since the set

$$I_{\alpha_0} = \{U \in \tau \mid U \cap U_{\alpha_0} = \emptyset\}$$

is an ideal (indeed a maximal ideal), $\mathcal{F}$ is a filter by virtue of Theorem 3. To show $\mathcal{F}$ is an ultrafilter, assume $\mathcal{F}^*$ is a filter which contains $\mathcal{F}$. Let V be an arbitrary element in $\mathcal{F}^*$. Either $V \cap U_{\alpha_0} = \emptyset$ or $V \cap U_{\alpha_0} \neq \emptyset$. If $V \cap U_{\alpha_0} = \emptyset$ then $X - V \in \mathcal{F} \subset \mathcal{F}^*$. But $X - V, V \in \mathcal{F}^*$ implies that $\emptyset - (X - V) \cap V \in \mathcal{F}^*$ which contradicts the fact that $\mathcal{F}^*$ is a filter. So $V \cap U_{\alpha_0} \neq \emptyset$. Since U_{α_0} is an element of the disjoint basis and is the smallest open set in τ, then $U_{\alpha_0} \subset V$ which means $\mathcal{F}^* \subset \mathcal{F}$. This shows that $\mathcal{F}$ is an ultrafilter.

In the next theorem, we will characterize the ultrafilters and maximal ideals in the Boolean ring τ.

Theorem 5. Let us consider the comp-topology τ as a Boolean ring. Then a proper subset I of τ is a maximal ideal if and only if the set $\mathcal{F} = \{X - U \mid U \in I\}$ is an ultrafilter.

Proof. Assume I is a maximal ideal. We show that $\mathcal{F} = \{X - U \mid U \in I\}$ is an ultrafilter. By virtue of Theorem 3, $\mathcal{F}$ is a filter. Assume $\mathcal{F}$ is contained in a filter $\mathcal{F}^*$. We define a subset I^* as $I^* = \{V \in \tau \mid X - V \in \mathcal{F}^*\}$. By Theorem 3, I^* is an ideal. Let U be an arbitrary element in I. Then $X - U \in \mathcal{F} \subset \mathcal{F}^*$, which implies that $U \in I^*$. Since I is a maximal ideal, so $I^* = I$. It is clear that $I^* = I$ implies that $\mathcal{F} = \mathcal{F}^*$ and $\mathcal{F}$ is an ultrafilter.

Now assume $\mathcal{F} = \{X - U \mid U \in I\}$ is an ultrafilter. We show that I is a maximal ideal. Let I^* be an ideal containing I. We define the set $\mathcal{F}^* = \{X - V \mid V \in I^*\}$. By Theorem 3, $\mathcal{F}^*$ is a filter and is containing the ultrafilter $\mathcal{F}$. So $\mathcal{F} = \mathcal{F}^*$. Let V be an arbitrary element in I^*. Then $X - V \in \mathcal{F}^* = \mathcal{F}$ which by the definition of $\mathcal{F}$ implies $V \in I$ and I is a maximal ideal.

If the index set A of the disjoint basis for τ is infinite, then in [3], it has been shown that τ has infinitely many maximal ideals. By Theorem 5, this is also true for the ultrafilters of the Boolean ring τ.

3. Topological Entropy. To evaluate the topological entropy of a comp-topology with respect to a homeomorphism, we must start with some basic definitions and properties.

Definition 1. An open covering $\mathcal{U}^*$ is said to be a refinement of an open covering $\mathcal{U}$ of a topological space X if every element of $\mathcal{U}^*$ is a subset of some element of $\mathcal{U}$ containing it.

It is clear that if X is a comp-topological space with $\beta = \{U_\alpha \mid \alpha \in A\}$ as the disjoint basis, then β refines every open cover of X.

The following definitions are taken from Adler, Konheim and McAndrew [1] and [4].

Definition 2. For any open cover $\mathcal{U}$ of X, we define $N(\mathcal{U})$ as the number of sets in a subcover of minimal cardinality. A subcover of a cover is minimal if no other subcover contains fewer members.

If X is a comp-topological space, then the disjoint basis $\beta = \{U_\alpha \mid \alpha \in A\}$ is an open covering with minimal cardinality. So $N(\beta) = \text{Card } A$.

Proposition 5. If $\mathcal{U}$ is an open cover of a comp-topological space with the disjoint basis $\beta = \{U_\alpha \mid \alpha \in A\}$, then $N(\mathcal{U}) \leq \text{Card } A$.

Proof. Since β refines $\mathcal{U}$, for any $U_\alpha \in \beta$ there is at least one element say $V_\alpha \in \mathcal{U}$ such that $U_\alpha \subset V_\alpha$ and $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$ is an open subcover of $\mathcal{U}$, possibly not a minimal. Thus, $N(\mathcal{U}) \leq N(\mathcal{V}) = \text{Card } A$.

Definition 3. For any two open covers $\mathcal{U}$ and $\mathcal{V}$, the set

$$\mathcal{U} \vee \mathcal{V} = \{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V}\}$$

defines their join. If X is a complementary topological space with the disjoint basis

$$\beta = \{U_\alpha \mid \alpha \in A\},$$

then for any open cover $\mathcal{U}$, the set

$$\mathcal{U} \vee \beta = \{U \cap U_\alpha \mid U \in \mathcal{U} \text{ and } U_\alpha \in \beta\} = \beta,$$

since $U \cap U_\alpha = \emptyset$ or $U \cap U_\alpha = U_\alpha$.

Definition 4. Let $f\colon X \to X$ be continuous and $\mathcal{U}$ an open cover of X. Let $f^{-1}(\mathcal{U})$ denote the open cover consisting of the inverse image of every element of $\mathcal{U}$; inductively define f^{-i} for all positive integers i.

Let the topological entropy of f with respect to $\mathcal{U}$, denoted by $\text{ent}(f,\mathcal{U})$, be defined by

$$\lim_{n\to\infty} n^{-1}\log(N((\mathcal{U})\vee f^{-1}(\mathcal{U})\vee\cdots\vee f^{-n+1}(\mathcal{U}))).$$

Theorem 6. If X is a compact comp-topological space with the disjoint basis

$$\beta=\{U_\alpha \mid \alpha\in A\},$$

then for any homeomorphism $h\colon X\to X$, $\text{ent}(h,\beta)=0$.

Proof. Since X is compact, then A is finite. By Theorem 3, for each fixed $i=1,\ldots,n$ and each element $h^{-i}(U_\alpha)\in h^{-i}(\beta)$, there exists an element $U_{\alpha_i}\in\beta$ such that

$$U_\alpha=h^{-i}(U_\alpha).$$

This shows that for $i\neq j$, $h^{-i}(U_\alpha)\cap h^{-j}(U_\alpha)$ is either $\emptyset$ or $U_{\alpha_i}\in\beta$. Therefore,

$$N(\beta\vee h^{-1}(\beta)\vee\cdots\vee h^{-n+1}(\beta))\leq \text{Card } A$$

and

$$\text{ent}(h,\beta)\leq\lim_{n\to\infty} n^{-1}\log(\text{Card } A)=0.$$

The next theorem shows that the topological entropy of h with respect to any open cover is also zero.

Theorem 7. If X is a compact comp-topological space with the finite basis

$$\beta=\{U_\alpha \mid \alpha\in A\},$$

then for any homeomorphism $h\colon X\to X$ and any open cover $\mathcal{U}$, $\text{ent}(h,\mathcal{U})=0$.

Proof. The family of sets

$$U\vee h^{-1}(\mathcal{U})\vee\cdots\vee h^{-n+1}(\mathcal{U})$$

is an open cover for X and it is clear that

$$N(\mathcal{U}\vee h^{-1}(\mathcal{U})\vee\cdots\vee h^{-n+1}(\mathcal{U}))\leq nN(\mathcal{U}).$$

By employing Proposition 1, $nN(\mathcal{U})\leq n$ Card A. Therefore,

$$\text{ent}(h,U)=\lim_{n\to\infty} n^{-1}\log(N(\mathcal{U}\vee h^{-1}(\mathcal{U})\vee\cdots\vee h^{-n+1}(\mathcal{U})))\leq\lim_{n\to\infty} n^{-1}\log(n\text{ Card } A)=0.$$

We used L'Hôpital's rule for finding this limit.

R. L. Adler, A. G. Konheim, and M. H. McAndrew [1] stated that the entropy $\mathrm{ent}(\phi)$ of a mapping ϕ is defined as the sup $\mathrm{ent}(\phi,\mathcal{U})$, where the supremum is taken over all open covers $\mathcal{U}$. Considering this definition and applying Theorem 7, we conclude that if $h\colon X \to X$ is a homeomorphism on a compact comp-topological space, then $\mathrm{ent}(h) = 0$.

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