INDEPENDENT RANDOM VARIABLES ON THE UNIT INTERVAL

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Let the probability of a subset of $[0,1]$ be given by its Lebesgue measure, i.e., the uniform distribution. In this paper we relate independent random variables, which are continuous functions, to space filling curves. In modern terminology, a random variable is a real valued measurable function defined on a probability space. A collection V of random variables is said to be an independent collection if, for any natural number n and for any collection of functions $\{f_1, f_2, \ldots, f_n\} \subset V$ and Borel subsets (equivalently, open intervals) $A_1, A_2, \ldots, A_n$ of $\mathbb{R}$, we have

$$\text{(1)} \qquad \Pr\left(\bigcap_{i=1}^{n}\{x \mid f_i(x) \in A_i\}\right) = \prod_{i=1}^{n} \Pr\left(\{x \mid f_i(x) \in A_i\}\right).$$

We are interested in considering random variables defined on the probability space consisting of the unit interval with the probability of a set given by its Lebesgue measure. A classical example of a collection of independent random variables defined on $[0,1]$ is that of the Rademacher functions $\{f_n(x)\}_{n=1}^{\infty}$ where

$$f_n(x) = 1, \text{ if } x \in [m/2^n, (m+1)/2^n)$$

with m even and

$$f_n(x) = -1, \text{ if } x \in [m/2^n, (m+1)/2^n)$$

with m odd.

Considerable work has been done studying general measurable functions which are independent on $[0,1]$ and on the intervals $[0,\infty)$ and $\mathbb{R} = (-\infty, \infty)$.

Many references to early work on this subject as well as the papers themselves, may be found in [4]. The possibility that such functions be continuous began with the observation that the coordinate functions of the Peano curve, which maps $[0,1]$ continuously into the unit square, are independent. (See [1] or [2] for recently discovered properties and a lucid description of the Peano curve.) Sierpinski [3] showed that if $x(t)$ and $y(t)$ are the coordinate functions of the Peano curve, then $f_n(t) = x(y^n(t))$, $n = 0, 1, 2, \ldots$ where $y^0(t) = t$, $y^n(t) = y(y^{n-1}(t))$ are independent and map $[0,1]$ onto $[0,1]^{\omega}$.

Most commonplace non-constant, continuous functions are not independent. In fact, if f_1 and f_2 are defined on $[0,1]$ and if there are intervals $[a,b]$ in the range of f_1 and $[c,d]$ in the range of f_2 so that $f_1^{-1}([a,b]) \cap f_2^{-1}([c,d]) = \emptyset$, then f_1 and f_2 are not independent. This is because

$$m(\{x \mid f_1(x) \in [a,b]\}) \cdot m(\{x \mid f_2(x) \in [c,d]\}) \neq 0$$

and

$$m(\{x \mid f_1(x) \in [a,b] \text{ and } f_2(x) \in [c,d]\}) = 0.$$

This suggests that, in order that two non-constant continuous functions be independent on $[0,1]$, the functions must be somewhat unusual. This is borne out by the theorem below.

Theorem 1. If f_1 and f_2 are continuous, non-constant independent functions defined on $[0,1]$ and if the range of f_1 is $[a,b]$ and the range of f_2 is $[c,d]$, then $F(x) = (f_1(x), f_2(x))$ is a continuous map from $[0,1]$ onto $[a,b] \times [c,d]$; that is, $F(x)$ is a "space-filling curve".

Proof. Suppose f_1 and f_2 are given as in the statement of the theorem and that (x_0, y_0) is any point in $[a,b] \times [c,d]$. Let (a',b') and (c',d') be any two open intervals with $x_0 \in (a',b')$ and $y_0 \in (c',d')$. Since f_1 and f_2 are continuous, $f_1^{-1}((a',b'))$ and $f_2^{-1}((c',d'))$ are non-empty open sets and, hence, are of positive Lebesgue measure. Since f_1 and f_2 are independent,

$$\begin{aligned} &m(\{x \mid f_1(x) \in (a',b') \text{ and } f_2(x) \in (c',d')\}) \\ &= m(\{x \mid f_1(x) \in (a',b')\}) \cdot m(\{x \mid f_2(x) \in (c',d')\}) > 0. \end{aligned}$$

Since this is true, there is a point t such that $F(t) = (f_1(t), f_2(t))$ belongs to $(a',b') \times (c',d')$. Since (a',b') and (c',d') are arbitrary intervals satisfying $x_0 \in (a',b')$ and $y_0 \in (c',d')$ and since $F(x)$ is continuous, it follows that there is $t_0 \in [0,1]$ such that $F(t_0) = (x_0, y_0)$. Then, (x_0, y_0) being an arbitrary point in $[a,b] \times [c,d]$, it follows that the range of F is $[a,b] \times [c,d]$ and thus, F is a "space-filling curve".

Clearly, the same result holds true if Lebesgue measure is replaced by any non-atomic measure whose support is an interval $[u,v]$. Also it is clear that if $\{f_n\}$ are finite or infinite sequence of independent random variables which are continuous on $[a,b]$, then

$$F(x) = (f_1(x), f_2(x), \dots)$$

is a "space filling curve," that is a continuous map from $[a,b]$ to $[u,v]^n$ or $[u,v]^\omega$.

References

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3. W. Sierpinski, "Remarque sur la Courbe Peannienne," *Wiad Matematyczne*, 42 (1936), 1.

4. H. Steinhaus, *Selected Papers*, Polish Scientific Publishers, Warsaw, 1985.