BASE NEGATIVE TEN

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Although it is not commonly known, a species similar to Homosapiens lives on planet X in a universe on the other side of a black hole. They use a base negative ten system of numeration. In order to exchange mathematical knowledge, there is a need for earthling translators who are familiar with the arithmetic algorithms of the Xlings. This article on base negative ten is being written in the hope that it will inspire some readers with an adventuresome spirit and a healthy mathematical curiosity to prepare for this post.

The base negative ten system of numeration employs ten symbols, none of which can be found on this keyboard. Hence, we shall utilize the numerals $0, 1, 2, \ldots, 9$ rather than those actually used by the Xlings.

The Xlings adhere to the following interuniversal principle of number bases:

INTERUNIVERSAL PRINCIPLE OF NUMBER BASES: In base b, the kth digit to the left of the "decimal" point indicates the number of b^{k-1}'s. The kth digit to the right of the decimal point indicates the number of b^{-k}'s.

So in base -10, the digits are used as follows:

$$\begin{array}{cccccccccc} \ldots & - & - & - & - & . & - & - & - & \ldots \\ \ldots & (-10)^3 & (-10)^2 & (-10)^1 & (-10)^0 & & (-10)^{-1} & (-10)^{-2} & (-10)^{-3} & \ldots \end{array}$$

Thus, the Xling's numeral "34" would be translated to $3(-10)+4$ or -26 on earth. Similarly, 124 in base negative ten represents our

$$1(-10)^2 + 2(-10) + 4 = 84.$$

Counting (that is, adding 1). The Xlings and the Earthlings both begin counting 1,2,3,4,5,6,7,8,9. However, while on earth, 1 more than 9 is 10, on planet X 1 more than 9 is (think about this before reading on).

Did you figure it out? $9 + 1 = 190$ in base negative ten, since 190 would translate to $1(100) + 9(-10) + 0$ or 10 on earth. After 190 comes 191, then $192, 193, \ldots, 198, 199, 180$.

And now one of the superior features of using base negative ten becomes evident. Whereas on earth we are encumbered with using a negative sign to denote negative numbers, no such sign is necessary on planet X. That is, all negative (as well as positive) numbers can be expressed in base negative ten without using the negative sign! For example, earth's -7 is equivalent to Xling's 13.

Addition. An addition problem that involves no regrouping, such as $34 + 45$ has the same answer in either base (here, 79). A striking difference between the two bases becomes evident however, when regrouping is required. Consider for example, the following base negative ten problem:

$$\begin{array}{r} 48 \\ +\ \underline{\ 2} \end{array}\ .$$

Since $8 + 2 = 190$ the reader might be tempted to write down the 0 and "carry the 19," which is merely postponing the dilemma, for one is then faced with adding $9 + 4$ in the second column. Indeed, the reader intent on pursuing this approach will soon find himself in a never-ending process of carrying. (Try it!) Instead, let us observe that in base ten,

$$[4(-10) + 8] + 2 = 4(-10) + (8 + 2) = 4(-10) + 10 = 3(-10)$$

which is simply 30 in base negative ten. That is,

$$\begin{array}{r} 3 \\ \cancel{4}8 \\ +\ \underline{\ 2} \\ 30 \end{array}\ .$$

In general, when regrouping is called for it is necessary to borrow rather than carry!

Comment on notation: The slash ("/") on the 4 and the 3 written above it indicate that after regrouping there are 3 negative tens.

Here is another example: $893 + 238$

$$\begin{array}{r} 78 \\ \cancel{8}\cancel{9}3 \\ +\ \underline{238} \\ 911 \end{array}\ .$$

Anytime you like, you can check your answer by translating all numerals to base ten. Here the corresponding earth problem would be $713 + 178 = 891$. Of course, to actually DO the problem by translating to base ten would be considered behavior unbecoming a prospective interpreter, and hence, forbidden! One final example: $8998 + 9757$

$$\begin{array}{r} 788 \\ \cancel{8}\cancel{9}\cancel{9}8 \\ +\ \underline{9757} \\ 196535 \end{array}\ .$$

Note the final carry of the "19", which incurs no difficulty here. $7 + 9 = 196$, so $7000 + 9000 = 196000$.

Another superior feature of the base negative ten system of numeration is that there is no need for elaborate rules to describe computation with "signed" numbers. The same algorithm applies to all cases. For example, the earth problem $74 + -25 = 49$ is equivalent to the Xling problem $134 + 35 = 169$.

Subtraction. Once again, when no regrouping is required, the Earthlings and the Xlings use precisely the same procedure to subtract. For example, in both bases the answer to $48 - 36$ is 12. However, when regrouping is required the negative base people don't borrow. Of course not! They carry!

Example: Compute $17 - 9$

$$\begin{array}{r} 2 \\ \cancel{1}7 \\ -\ \underline{9} \\ 28 \end{array} \quad .$$

In base ten:

$$\begin{aligned} [1(-10) + 7] - 9 &= 1(-10) + (7 - 9) = 1(-10) + (-10 + 10) + (7 - 9) \\ &= 2(-10) + (17 - 9) = 2(-10) + 8 \end{aligned}$$

which is 28 in base negative ten.

Observe that a problem such as $-4 - (-5)$ in base 10 which many an earthling teacher must struggle to explain is equivalent to the base negative ten problem $16 - 15$, which is clearly 1.

A few more examples: $34 - 298$; $2 - 444$; and $0 - 50929$.

$$\begin{array}{r} 114 \\ \cancel{0}\cancel{3}4 \\ -\ \underline{298} \\ 1956 \end{array} \qquad \begin{array}{r} 111 \\ \cancel{0}\cancel{0}2 \\ -\ \underline{444} \\ 1778 \end{array} \qquad \begin{array}{r} 111 \\ 0\cancel{0}\cancel{0}\cancel{0}0 \\ -\ \underline{50929} \\ 151291 \end{array}$$

Multiplication. In base negative ten, $5 \times 7 = 175$. (It might be helpful to us earthlings to think: $5 \times 7 = 100 - 70 + 5$). For your convenience, below is a table of the basic multiplication facts in base negative ten.

×	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	190	192	194	196	198
3	3	6	9	192	195	198	181	184	187
4	4	8	192	196	180	184	188	172	176
5	5	190	195	180	185	170	175	160	165
6	6	192	198	184	170	176	162	168	154
7	7	194	181	188	175	162	169	156	143
8	8	196	184	172	160	168	156	144	132
9	9	198	187	176	165	154	143	132	121.

Before considering an algorithm for multiplication, note that in any base which utilizes, say, the digits 2, 3, and 6, $20 \times 300 = (2 \times 10) \times (3 \times 100) = (2 \times 3) \times (10 \times 100) = 6 \times 1000 = 6000$.

Since $5 \times 7 = 175$ in base negative ten, it follows that $5 \times 70 = 1750$, and $5 \times 700 = 17500$. Number properties are independent of systems of numeration.

Now, to multiply 5×76, the Xlings multiply 5×6 and add this to 5×70.

$$\begin{array}{rl}
76 & \\
\times\ \underline{\ 5} & \\
170 & (5 \times 6) \\
+\ \underline{1750} & (5 \times 70) \\
1720 & (5 \times 76)
\end{array}$$

Observe that -2×-3 in base ten is equivalent to 18×17 in base negative ten and:

$$\begin{array}{rl}
18 & \\
\times\ \underline{17} & \\
156 & (7 \times 8) \\
70 & (7 \times 10) \\
80 & (10 \times 8) \\
+\ \underline{100} & (10 \times 10) \\
6 &
\end{array}$$

Other examples:

$$\begin{array}{r} 9 \\ \times\ \underline{\ 27} \\ 143 \\ +\ \underline{1980} \\ 1923 \\ \\ \\ \end{array} \qquad\qquad \begin{array}{r} 394 \\ \times\ \underline{\ 8} \\ 172 \\ 1320 \\ +\ \underline{18400} \\ 19892 \\ \\ \end{array} \qquad\qquad \begin{array}{r} 74 \\ \times\ \underline{\ 35} \\ 180 \\ 1750 \\ 1920 \\ +\ \underline{18100} \\ 19750 \end{array}$$

Of course there exist more efficient algorithms for multiplication. The reader is encouraged to discover one.

Division. (one digit divisors): When performing the long division problem

$$b \mid a \;\;, \text{ where } b > 0$$

the usual strategy is to keep subtracting multiples of b until you obtain a remainder r such that $0 \le r < b$. The quotient q is the number of multiples of b that were subtracted. Thus, $a = qb + r$ where $0 \le r < b$. In the base ten examples below, the numbers in parentheses are the multiples being subtracted. To focus on the concepts involved when dividing, we will dispense with efficiency.

Base ten

$$\begin{array}{rrr} 5 & \overline{)427} & \\ - & \underline{100} & (20) \\ & 327 & \\ - & \underline{250} & (50) \\ & 77 & \\ - & \underline{75} & (15) \\ & 2 & \overline{(85)} \end{array}$$

Thus, $427 = 85 \times 5 + 2$.

It is not necessary to repeatedly subtract POSITIVE multiples of the divisor.

Base ten

$$\begin{array}{rrr} 5 & \overline{)427} & \\ - & \underline{500} & (100) \\ & -73 & \\ - & \underline{-75} & (-15) \\ & 2 & \overline{(85)} \end{array}$$

Thus, $427 = 85 \times 5 + 2$. Also,

$$\begin{array}{rrr} 7 & \overline{)-132} & \\ - & \underline{-70} & (-10) \\ & -62 & \\ - & \underline{-63} & (-9) \\ & 1 & \overline{(-19)} \end{array}$$

Thus, $-132 = -19 \times 7 + 1$.

In the first example, we overshot 0, so we compensated by subtracting a negative multiple of 5.

This procedure for dividing is, of course, the same independent of the base. It is not necessary to use it efficiently, as is illustrated below in the first example of long division in *base negative ten.* Once again, the numbers in parentheses indicate the multiple of the divisor being subtracted.

$$\begin{array}{rrr} 5 & \overline{)237} & \\ - & \underline{165} & (9) \\ & 272 & \\ - & \underline{165} & (9) \\ & 127 & \\ - & \underline{165} & (9) \\ & 162 & \\ - & \underline{160} & (8) \\ & 2 & \overline{(175)} \end{array}$$

Thus, $237 = 175 \times 5 + 2$. Also,

$$\begin{array}{rrr} 8 & \overline{)9874} & \\ - & \underline{8000} & (1000) \\ & 1874 & \\ - & \underline{1720} & (40) \\ & 154 & \\ - & \underline{168} & \underline{(6)} \\ & 6 & (1046) \end{array}$$

Thus, $9874 = 1046 \times 8 + 6$.

Again, there exist more efficient algorithms for dividing, and the reader is encouraged to find one.

Although we plan to postpone discussion of division by a multidigited divisor until the first publication of the Interuniversal Primer, we wish to note that when the Xlings divide by a two digit number (or any divisor with an even number of digits), they require a remainder which is greater than or equal to zero and less than the additive inverse of the divisor.

As a final note, we Earthlings have already benefitted from our initial study of base negative ten arithmetic by obtaining an alternate method of summing geometric series in which the ratio between two consecutive terms is negative.

Recall that in base 10, one method of finding the sum

$$.3 + .03 + .003 + .0003 + \cdots = .3333\ldots$$

is to argue as follows. Let $S = .3333\ldots$. Then $10S = 3.3333\ldots$. Subtracting, S from $10S$ we obtain, $9S = 3$. Hence, $S = 3/9 = 1/3$.

Now consider the problem,

$$1 - .1 + .01 - .001 + .0001 - .00001 + \cdots.$$

In base negative ten this is equivalent to $1.111\ldots$ and can be summed as follows. Let $S = 1.11111\ldots$. Then $10S = 11.11111\ldots$. Subtracting S from $10S$ we obtain $(10-1)S = 10$. In base ten, this is $[1(-10) - 1]S = 1(-10)$ or $-11S = -10$. So, $S = 10/11$.

Note that a similar technique could be used to find the sum of any geometric series for which the ratio between successive terms is integral. Indeed, this technique may even be applicable when the ratio isn't an integer. The interested reader will find a treatise on this subject in the second edition of the Interuniversal Primer.

Some Topics for Further Investigation. Investigate:

1. the long division algorithm when the divisor has more than 1 digit.
2. the algorithms when applied to base negative ten fractions and "decimals".
3. ordinary algebra in base negative ten.
4. unit fractional bases and negative number bases other than negative ten.

References

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