A NOTE ON MODULE HOMOMORPHISMS

John Koker

University of Wisconsin - Oshkosh

During a recent course in ring theory, I asked a student of mine to find examples of a ring and modules to show that the two conditions for a function to be a module homomorphism are independent of each other. Let R be a (not necessarily commutative) ring and M and N be left R-modules. A function $f: M \to N$ is said to be an R-module homomorphism if

(1) $f(x+y) = f(x) + f(y)$ for all $x, y \in M$ and

(2) $f(rx) = rf(x)$ for all $x \in M$, $r \in R$.

Let R be a noncommutative ring and fix $r \in R$ with r not in the center of R. Then $f: R \to R$ defined by $f(s) = rs$ is a function from the left R-module R to itself which satisfies condition (1). However, due to the noncommutativity, there exists $s, t \in R$ such that $f(ts) = rts \neq trs = tf(s)$. Thus (2) fails.

Likewise, we can find an example to satisfy (2) but not (1). Let $R = \mathbb{Z}_2$ and let $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Define $f: M \to M$ by $f(0,0) = (0,0)$ and $f(x,y) = (1,1)$ if $(x,y) \neq (0,0)$. It is straight forward to check that (2) is satisfied. But $f[(1,0)+(0,1)] = f(1,1) = (1,1) \neq f(1,0) + f(0,1)$. Hence, (1) fails.

One question this note addresses is when does (2) imply (1)? The converse is more interesting if (2) implies (1), what can be said about M? Recall that M is a cyclic left R-module if there exists $x \in M$ with $Rx = M$.

Lemma 1. Suppose that R is a ring and N is any left R-module. If M is a cyclic left R-module and $f: M \to N$ is a function which satisfies (2), then f satisfies (1).

Proof. Choose $x \in M$ with $M = Rx$. Let $x_1, x_2 \in M$. Then there exists $r_1, r_2 \in R$ with $x_1 = r_1 x$ and $x_2 = r_2 x$. Thus, $f(x_1 + x_2) = f(r_1 x + r_2 x) = f[(r_1 + r_2)x] = (r_1 + r_2) f(x) = r_1 f(x) + r_2 f(x) = f(r_1 x) + f(r_2 x) = f(x_1) + f(x_2)$.

Thus, M being cyclic is a sufficient condition for (2) to imply (1). We now exhibit necessary conditions on M. In other words, if $f: M \to N$ is a function in which (2) implies (1), what can be said about M?

Suppose that F is a field and V is a vector space over F. The definition of a linear transformation from V to V is the same as for module homomorphisms. Finding examples to show the independence of (1) and (2) is a common exercise for linear algebra students. Working in this situation aids in answering the above question.

Theorem 2. Suppose that V is a vector space over F. The dimension of V is 1 if and only if any function $f: V \to V$ which satisfies (2) also satisfies (1).

Proof. If V is of dimension 1, then Lemma 1 provides the result. To prove the converse, fix $0 \neq \alpha \in V$. Define a map $f: V \to V$ by $f(\beta) = \beta$ if $\beta \in F\alpha$ and $f(\beta) = 0$ if $\beta \notin F\alpha$ for all $\beta \in V$. First it is shown that for all $x \in F$ and $\beta \in V$, $f(x\beta) = xf(\beta)$.

Case 1. Suppose that $x\beta \in F\alpha$. If $x = 0$, then $f(x\beta) = 0 = xf(\beta)$. If $x \neq 0$, then $x^{-1} \in F$. Thus, $\beta = x^{-1}(x\beta) \in F\alpha$ and so $f(x\beta) = x\beta = xf(\beta)$.

Case 2. Suppose that $x\beta \notin F\alpha$. Then $\beta \notin F\alpha$ since $F\alpha$ is closed under scalar multiplication. Therefore $f(x\beta) = 0 = xf(\beta)$. By hypothesis, f is a linear transformation.

Claim. $V = F\alpha$. If $\beta, \gamma \in V$ with $0 \neq \beta + \gamma \in F\alpha$, then $0 \neq \beta + \gamma = f(\beta + \gamma) = f(\beta) + f(\gamma)$. Therefore $f(\beta) \neq 0$ or $f(\gamma) \neq 0$. Assume that $f(\beta) \neq 0$. Then $\beta \in F\alpha$. Thus, it follows that $\gamma = (\gamma + \beta) - \beta \in F\alpha$. Thus, $0 \neq \beta + \gamma \in F\alpha$ implies that $\beta \in F\alpha$ and $\gamma \in F\alpha$.

Now, for $\delta \in V$, $0 \neq \delta + (-\delta + \alpha) \in F\alpha$. Thus, $\delta \in F\alpha$. Hence, $V = F\alpha$ and $\dim(V) = 1$.

This proof relied upon the fact that each non-zero element of F had an inverse. Thus, the following generalization is obtained. Recall that a simple left R-module $M \neq 0$ is a module with 0 and M being its only submodules.

Theorem 3. Let R be a division ring (not necessarily commutative), and let M be a left R-module. Then M is simple if and only if every function $f: M \to M$ which satisfies (2) satisfies (1).

One may wonder if this result generalizes to rings for which some of the elements are not units? A natural place to start is to consider domains. Again, it is not required for these to be commutative. If R is a domain and M is a left R-module, we say that M is torsion-free if its torsion submodule $\{m \in M \mid rm = 0 \text{ for some } 0 \neq r \in R\}$ is zero. Using these ideas, the following is obtained.

Theorem 4. Suppose that R is a domain and that M is a torsion-free left R-module which has a simple submodule. Then M is simple if and only if every function $f: M \to M$ which satisfies (2) satisfies (1).

Proof. If M is simple, then M is cyclic and so Lemma 1 gives the result.

Conversely, assume that any function $f\colon M \to M$ satisfying (2) satisfies (1). Since $1 \in R$, M has a simple submodule of the form Rx for some $x \in M$. It needs to be shown that $Rx = M$. Define $f\colon M \to M$ by $f(a) = a$ if $a \in Rx$ and $f(a) = 0$ if $a \notin Rx$ for all $a \in M$.

Let $m \in M$ and $r \in R$. If $rm \notin Rx$, then $m \notin Rx$. Thus $f(rm) = 0 = rf(m)$. On the other hand, suppose $rm \in Rx$. If $m \in Rx$, then $f(rm) = rm = rf(m)$. Finally, suppose that $m \notin Rx$. Consider the set $(m : Rx) = \{s \in R \mid sm \in Rx\}$. This set is a left ideal of R and thus $(m : Rx)x$ is a submodule of Rx. Since Rx is simple, $(m : Rx)x = 0$ or $(m : Rx)x = Rx$. If $(m : Rx)x = Rx$, then there exists $s \in (m : Rx)$ with $sx = x$. Thus, $(s-1)x = 0$. However, M torsion-free implies that $s = 1 \in (m : Rx)$. This in turn implies that $m \in Rx$ which is a contradiciton. Therefore $(m : Rx) = 0$. The hypothesis $rm \in Rx$ and $m \notin Rx$ implies that $r = 0$. This says that $f(rm) = 0 = rf(m)$. Therefore f satisfies (2) and so it is a homomorphism.

The proof is completed by showing that $M = Rx$, which is similar to the proof of Theorem 2.

This is a natural generalization of Theorem 2 since every torsion-free R-module over a domain R can be embedded in a vector space over Q where Q is a quotient field of R [1, Lemma 4.31].

Reference

1. J. J. Rotman, *An Introduction to Homological Algebra,* Academic Press, New York, 1979.