EVALUATION OF A FAMILY OF SUMMATIONS

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The purpose of this paper is to evaluate

$$S = S(p) = \sum_{n=1}^{2p-1} \left(2^{n-1} \prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n} + \frac{n\pi}{2p} \right) \right) \tag{1}$$

where p is any positive integer. When $n = 1$ the first term in the sum is taken to be 1 since empty products are assumed to be 1.

<u>Lemma</u>. If $\sin\theta \neq 0$, then

$$2^{n-1} \prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n} + \theta \right) = \frac{\sin n\theta}{\sin\theta}.$$

<u>Proof</u>. Since the nth roots of unity are

$$e^{\frac{2k\pi i}{n}}$$

for $k = 0, 1, \ldots, n-1$,

$$x^n - 1 = (x-1) \prod_{k=1}^{n-1} \left(x - e^{\frac{2k\pi i}{n}} \right) . \tag{2}$$

Putting $x = e^{2\theta i}$ in identity (2), yields

$$e^{2n\theta i} - 1 = (e^{2\theta i} - 1) \prod_{k=1}^{n-1} \left(e^{2\theta i} - e^{\frac{2k\pi i}{n}}\right) .$$

Therefore, for $\sin\theta \neq 0$,

$$\frac{\sin n\theta}{\sin\theta} = e^{-(n-1)\theta i} \prod_{k=1}^{n-1} \left(e^{2\theta i} - e^{\frac{2k\pi i}{n}}\right) . \tag{3}$$

Now,

$$\begin{aligned} e^{2\theta i} - e^{\frac{2k\pi i}{n}} &= e^{\left(\theta+\frac{k\pi}{n}\right)i}\left(e^{\left(\theta-\frac{k\pi}{n}\right)i} - e^{-\left(\theta-\frac{k\pi}{n}\right)i}\right) , \\ &= 2ie^{\left(\theta+\frac{k\pi}{n}\right)i}\sin\left(\theta - \frac{k\pi}{n}\right) . \end{aligned}$$

So,

$$\begin{aligned} \prod_{k=1}^{n-1}\left(e^{2\theta i} - e^{\frac{2k\pi i}{n}}\right) &= 2^{n-1} i^{n-1} e^{(n-1)\theta i + \frac{(n-1)\pi i}{2}} \prod_{k=1}^{n-1} \sin\left(\theta - \frac{k\pi}{n}\right) \\ &= 2^{n-1} e^{(n-1)(\theta+\pi)i} \prod_{k=1}^{n-1} \sin\left(\theta - \frac{k\pi}{n}\right) . \end{aligned} \tag{4}$$

Substituting equation (4) into (3) and simplifying yields

$$\frac{\sin n\theta}{\sin\theta} = (-1)^{n-1} 2^{n-1} \prod_{k=1}^{n-1} \sin\left(\theta - \frac{k\pi}{n}\right) . \tag{5}$$

The desired result follows from (5) when θ is replaced with $-\theta$.

Theorem. If p is a positive integer, then

$$S = \sum_{n=1}^{2p-1} \frac{\sin\left(\frac{n^2\pi}{2p}\right)}{\sin\left(\frac{n\pi}{2p}\right)} = p \ .$$

Proof. For $\sin\theta \neq 0$, divide both sides of the identity

$$\sum_{n=1}^{2p-1} \sin n\theta = \sum_{j=1}^{p} \sin(2j-1)\theta + \sum_{j=1}^{p-1} \sin 2j\theta$$

by $\sin\theta$ and use Lagrange's trigonometric identity to get

$$\sum_{n=1}^{2p-1} \frac{\sin n\theta}{\sin\theta} = \sum_{j=1}^{p} \frac{\sin(2j-1)\theta}{\sin\theta} + \sum_{j=1}^{p-1} \frac{\sin 2j\theta}{\sin\theta} \ ,$$

$$= \sum_{j=1}^{p} \left(1 + 2\sum_{k=1}^{j-1} \cos 2k\theta\right) + 2\sum_{j=1}^{p-1}\sum_{k=1}^{j} \cos(2k-1)\theta \ .$$

In particular, for $\theta = \frac{n\pi}{2p}$,

$$S = \sum_{n=1}^{2p-1} \frac{\sin\frac{n^2\pi}{2p}}{\sin\frac{n\pi}{2p}} = p + 2\sum_{j=1}^{p}\sum_{k=1}^{j-1} \cos\frac{2kn\pi}{2p} + 2\sum_{j=1}^{p-1}\sum_{k=1}^{j} \cos\frac{(2k-1)n\pi}{2p}$$

$$= p + 2\sum_{j=1}^{p}\sum_{k=1}^{p-1} \cos\frac{k(2j-1)\pi}{p} \ . \tag{6}$$

It is known that

$$\sum_{k=1}^{p-1} \cos k\phi = (-1 + \sin p\phi \cot \frac{\phi}{2} - \cos p\phi)/2 \ .$$

So, if $p\phi = (2j-1)\pi$ where j is an integer, then

$$\sum_{k=1}^{p-1} \cos k\phi = 0 \ .$$

Therefore,

$$\sum_{k=1}^{p-1} \cos \frac{k(2j-1)\pi}{p} = 0$$

and so (6) yields $S = p$.