AN APPLICATION OF THE UNIQUE CONTINUATION PROPERTY TO THE COMPUTATION OF A FOURIER TRANSFORM

Joseph Bennish

California State University, Long Beach

In this paper we present another method for computing the Fourier transform of the Gaussian function $G(x) = \exp(-x^2)$. The idea is that its Fourier transform

$$F(\xi) = \int_{-\infty}^{\infty} e^{-x^2} e^{-ix\xi} dx \ ,$$

can be continued analytically to the entire complex plane and is easily computed along the imaginary axis. This computation is a striking example of the power of the unique continuation property which states that two functions analytic in a connected domain and agreeing on a set containing a limit point must agree on the entire domain (see, e.g., [2, p. 226]).

The analyticity of $F(z)$, $z = \xi + i\eta$, follows from differentiating under the integral sign. Indeed, this is permitted since the integrand is absolutely integrable and its partial derivatives with respect to ξ and η are absolutely integrable uniformly for $|z|$ bounded (see, e.g., [1, Theorem 53.5]). Computing $F(z)$ along the imaginary axis we get, upon making the change of variable $x = y + \eta/2$,

$$\begin{aligned} F(i\eta) &= \int_{-\infty}^{\infty} e^{-x^2} e^{x\eta} dx \\ &= \int_{-\infty}^{\infty} e^{-y^2} dy \exp(\eta^2/4) = \sqrt{\pi}\exp(\eta^2/4) \ . \end{aligned}$$

The evaluation of $I = \int \exp(-x^2)dx = \sqrt{\pi}$ is well-known and is included for completeness.

Changing variables to polar coordinates gives

$$I^2 = \int\int e^{-(x^2+y^2)} dxdy = \int_0^{2\pi} \int_0^{\infty} re^{-r^2} drd\theta = \pi \ .$$

The function $\sqrt{\pi}\exp(\eta^2/4)$ is the restriction of the analytic function $H(z) = \sqrt{\pi}\exp(-z^2/4)$ to the imaginary axis. The functions $H(z)$ and $F(z)$ agree on the imaginary axis, so, by the unique continuation property, these functions are identical for all z. Restricting to the real axis we obtain the Fourier transform of the Gaussian: $F(\xi) = \sqrt{\pi}\exp(-\xi^2/4)$.

References

1. T. W. Korner, *Fourier Analysis,* Cambridge University Press, Cambridge, 1988.
2. W. Rudin, *Real and Complex Analysis,* 2nd ed., McGraw-Hill Book Company, New York, 1974.