SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

33. [1991, 93] *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

If A, B, and C are the angles of a triangle, prove that

$$\cot A + \cot B + \cot C \geq \sqrt{3}\ .$$

Under what conditions will equality hold?

Solution I by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

It is known that

$$\cot A + \cot B + \cot C \geq \frac{(a^2+b^2+c^2)(a+b+c)\sqrt{3}}{9abc}$$

(where a is the length of the side of triangle ABC opposite vertex A, b is the length of the side of triangle ABC opposite vertex B, and c is the length of the side of triangle ABC opposite vertex C), with equality holding if and only if the triangle is equilateral. [See the Solution to Problem E1861 on pp. 724–725 of the June–July 1967 issue of *The American Mathematical Monthly.*]

By the Arithmetic Mean-Geometric Mean Inequality

$$\frac{a^2+b^2+c^2}{3} \geq \sqrt[3]{a^2b^2c^2} = (abc)^{\frac{2}{3}}$$

and

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc} = (abc)^{\frac{1}{3}}$$

with equality holding if and only if $a = b = c$. Thus

$$\frac{a^2+b^2+c^2}{3} \cdot \frac{a+b+c}{3} \geq abc$$

so

$$\frac{(a^2+b^2+c^2)(a+b+c)\sqrt{3}}{9abc} \geq \sqrt{3}\ ,$$

with equality holding if and only if $a = b = c$.

It follows that

$$\cot A + \cot B + \cot C \geq \sqrt{3}$$

with equality holding if and only if triangle ABC is equilateral.

Solution II by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

Our solution will use the following known results.

(1) $$\cot A + \cot B + \cot C \geq \frac{3}{5}(\csc A + \csc B + \csc C) - \frac{1}{5}\sqrt{3}$$

with equality if and only if the triangle is equilateral.

(2) $$\csc A + \csc B + \csc C \geq 2\sqrt{3}$$

with equality if and only if the triangle is equilateral.

[For a proof of (1) see the Solution to Problem E2323 on pp. 1040-1041 of the November 1972 issue of *The American Mathematical Monthly* and for a proof of (2) see the Solution to Problem E1861 on pp. 724–725 of the June–July 1967 issue of *The American Mathematical Monthly.*]

It follows from (1) and (2) that

$$\cot A + \cot B + \cot C \geq \frac{3}{5}(2\sqrt{3}) - \frac{1}{5}\sqrt{3} = \sqrt{3}$$

with equality if and only if the triangle is equilateral.

Solution III by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

Let ω denote the Brocard angle of triangle ABC. Then

(1) $$\cot \omega = \cot A + \cot B + \cot C$$

and

(2) $$\omega \leq \frac{\pi}{6}$$

with equality in (2) if and only if triangle ABC is an equilateral triangle. [See pp. 174–175 of Davis; *Modern College Geometry;* Addison-Wesley Publishing Company, Inc.; Reading, Massachusetts; 1957.] Since cotangent is a continuous decreasing function in $(0, \pi)$,

$$\cot \omega \geq \cot \frac{\pi}{6} \ .$$

Thus

$$\cot A + \cot B + \cot C = \cot \omega \geq \cot \frac{\pi}{6} = \sqrt{3} \ .$$

Also equality holds if and only if triangle ABC is equilateral.

Solution IV by Russell Euler, Northwest Missouri State University, Maryville, Missouri and Kandasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin (independently).

$$\begin{aligned}
\cot A + \cot B &= \frac{\cos A}{\sin A} + \frac{\cos B}{\sin B} \\
&= \frac{\cos A \sin B + \sin A \cos B}{\sin A \sin B} \\
&= \frac{\sin(A+B)}{\sin A \sin B} \\
&= \frac{\sin(\pi - C)}{(\cos(A-B) - \cos(A+B))/2} \\
&= \frac{2 \sin C}{\cos(A-B) - \cos(\pi - C)} \\
&= \frac{2 \sin C}{\cos(A-B) + \cos C} \\
&\geq \frac{2 \sin C}{1 + \cos C} \\
&= 2 \tan \frac{C}{2} \ .
\end{aligned}$$

Therefore,

$$\begin{aligned}\cot A + \cot B + \cot C &\geq 2\tan\frac{C}{2} + \cot C \\ &= 2\tan\frac{C}{2} + \frac{1}{\tan 2\left(\frac{C}{2}\right)} \\ &= 2\tan\frac{C}{2} + \frac{1-\tan^2\frac{C}{2}}{2\tan\frac{C}{2}} \\ &= \frac{3\tan^2\frac{C}{2}+1}{2\tan\frac{C}{2}}\ .\end{aligned}$$

Hence, since $a^2 + b^2 \geq 2ab$ for all real numbers a and b,

$$\cot A + \cot B + \cot C \geq \frac{2\sqrt{3}\tan\frac{C}{2}}{2\tan\frac{C}{2}} = \sqrt{3}\ .$$

Clearly, equality will hold iff $\sqrt{3}\tan\frac{C}{2} = 1$ and $\cos(A-B) = 1$. This system has

$$A = B = C = \frac{\pi}{3}$$

as its solution.

Solution V by Russell Euler, Northwest Missouri State University, Maryville, Missouri and Richard Edmundson (junior), University of Texas-Pan American, Edinburg, Texas (independently).

For any function f such that $f''(x) > 0$ for all x in the domain of f,

$$f\left(\frac{x+y+z}{3}\right) \leq \frac{f(x)+f(y)+f(z)}{3}\ .$$

So, for $f(x) = \cot x$ on $(0, \frac{\pi}{2})$,

$$\frac{\cot A + \cot B + \cot C}{3} \geq \cot\left(\frac{A+B+C}{3}\right)\ .$$

That is,

$$\cot A + \cot B + \cot C \geq 3 \cot \frac{\pi}{3} = \sqrt{3} \ ,$$

and equality holds only for equilateral triangles.

Solution VI by Russell Euler, Northwest Missouri State University, Maryville, Missouri and Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico (independently).

Since $A + B + C = \pi$, it suffices to minimize

$$f(A, B) = \cot A + \cot B + \cot(\pi - A - B) \ .$$

The critical values are the solutions of the system

$$\frac{\partial f}{\partial A} = -\csc^2 A + \csc^2(\pi - A - B) = 0$$

$$\frac{\partial f}{\partial B} = -\csc^2 B + \csc^2(\pi - A - B) = 0 \ .$$

Therefore, $\csc^2 A = \csc^2 B = \csc^2 C$ and so $A = B = C = \frac{\pi}{3}$. It is easy to show that

$$\frac{\partial^2 f(\frac{\pi}{3}, \frac{\pi}{3})}{\partial A^2} \cdot \frac{\partial^2 f(\frac{\pi}{3}, \frac{\pi}{3})}{\partial B^2} - \left(\frac{\partial^2 f(\frac{\pi}{3}, \frac{\pi}{3})}{\partial A \partial B} \right)^2 = \frac{64}{9} > 0$$

and

$$\frac{\partial^2 f(\frac{\pi}{3}, \frac{\pi}{3})}{\partial A^2} = \frac{16}{3\sqrt{3}} > 0 \ .$$

Hence, f has a minimum value of $\sqrt{3}$ at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and the desired result follows with equality holding only for equilateral triangles.

Also solved by N.J. Kuenzi, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

Bob Prielipp and a referee noted that the inequality appears in Bottema et. al., *Geometric Inequalities*, Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1968, 2.38, pp. 28–29.

34. [1991, 93] *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

Let m and k be positive integers and $1 \le k \le m$. Evaluate

$$\sum_{\substack{n_1+2n_2+\cdots+mn_m=m \\ n_1+n_2+\cdots+n_m=k}} \frac{m!}{(1!)^{n_1} n_1! \cdots (m!)^{n_m} n_m!} ,$$

where n_1, n_2, …, n_m are non-negative integers.

Solution by the proposers. Let t be fixed. It follows by an easy induction on m that

$$\frac{d^m}{dx^m} e^{t(e^x-1)} = e^{t(e^x-1)} \sum_{k=1}^{m} t^k e^{kx} \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \tag{1}$$

for any positive integer m. Here, $\{\cdot\}$ denotes a Stirling number of the 2nd kind. Also, Faà di Bruno's formula says that if $f(x)$ and $g(x)$ are functions for which all the necessary derivatives are defined and m is a positive integer, then

$$\frac{d^m}{dx^m} f(g(x)) = \sum_{n_1+2n_2+\cdots+mn_m=m} \frac{m!}{n_1! \cdots n_m!} \left(\frac{d^{n_1+\cdots+n_m}}{dx^{n_1+\cdots+n_m}} f \right)(g(x))$$

$$\cdot \left(\frac{\frac{d}{dx} g(x)}{1!} \right)^{n_1} \cdots \left(\frac{\frac{d^m}{dx^m} g(x)}{m!} \right)^{n_m}$$

where n_1, n_2, …, n_m are non-negative integers. Letting $f(x) = e^{tx}$ and $g(x) = e^x - 1$ and simplifying gives

$$\frac{d^m}{dx^m} e^{t(e^x-1)} = e^{t(e^x-1)} \sum_{k=1}^{m} t^k e^{kx} \sum_{\substack{n_1+2n_2+\cdots+mn_m=m \\ n_1+n_2+\cdots+n_m=k}} \frac{m!}{(1!)^{n_1} n_1! \cdots (m!)^{n_m} n_m!} , \tag{2}$$

where n_1, n_2, …, n_m are non-negative integers. Equating the right-hand sides of (1) and (2) and using the fact that the polynomials t, t^2, …, t^m are linearly independent, it follows that the expression we want to evaluate is

$$\left\{ \begin{matrix} m \\ k \end{matrix} \right\} .$$

35. [1991, 93] *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Let F_n denote the nth Fibonacci number ($F_1 = F_2 = 1$ and $F_n = F_{n-2} + F_{n-1}$ for $n \geq 3$) and let L_n denote the nth Lucas number ($L_1 = 1$, $L_2 = 3$ and $L_n = L_{n-2} + L_{n-1}$ for $n \geq 3$). Express L_n^2 as a polynomial in F_n.

Solution I by Alex Necochea, University of Texas-Pan American, Edinburg, Texas; Russell Euler, Northwest Missouri State University, Maryville, Missouri; and Miguel Paredes, University of Texas-Pan American, Edinburg, Texas (independently).

Let

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2} \ .$$

Then $\alpha \cdot \beta = -1$. Also, the Binet forms for Fibonacci and Lucas numbers are respectively

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{and} \quad L_n = \alpha^n + \beta^n$$

for $n \geq 1$. Then,

$$\begin{aligned} L_n^2 &= (\alpha^n + \beta^n)^2 \\ &= \alpha^{2n} + 2\alpha^n\beta^n + \beta^{2n} \\ &= \alpha^{2n} + 2 \cdot (-1)^n + \beta^{2n} \\ &= \alpha^{2n} - 2 \cdot (-1)^n + \beta^{2n} + 4 \cdot (-1)^n \\ &= (\alpha^n - \beta^n)^2 + 4 \cdot (-1)^n \\ &= 5 \cdot \left(\frac{\alpha^n - \beta^n}{\sqrt{5}} \right)^2 + 4 \cdot (-1)^n \\ &= 5 \cdot F_n^2 + 4 \cdot (-1)^n \ . \end{aligned}$$

Therefore, L_n^2 can be written as a polynomial in F_n as

$$L_n^2 = 5F_n^2 + 4 \cdot (-1)^n \ .$$

Solution II by Joseph E. Chance, University of Texas-Pan American, Edinburg, Texas.

The following two identities are standard results for Lucas and Fibonacci numbers:

$$L_n = F_{n+1} + F_{n-1} \ , \quad n \geq 2$$

$$F_{n-1}F_{n+1} = F_n^2 + (-1)^n \ , \quad n \geq 2 \ .$$

With these two identites the required result is derived. For $n \geq 2$,

$$\begin{aligned} L_n^2 - F_n^2 &= (F_{n+1} + F_{n-1})^2 - (F_{n+1} - F_{n-1})^2 \\ &= 4F_{n+1}F_{n-1} \\ &= 4\big(F_n^2 + (-1)^n\big) \ . \end{aligned}$$

Thus,

$$L_n^2 = 5F_n^2 + 4(-1)^n$$

and this identity also holds for $n = 1$.

Also solved by W.F. Wheatley III, Hazlehurst, Mississippi and Alex Necochea, University of Texas-Pan American, Edinburg, Texas (another solution).

Gerald E. Burgum, editor of *The Fibonacci Quarterly*, noted that

$$L_n^2 = 5F_n^2 + 4(-1)^n$$

is equation I_{12} in V.E. Hoggatt, Jr., *Fibonacci and Lucas Numbers*, Houghton Mifflin, 1969.

36. [1991, 94] *Proposed by James Taylor, Central Missouri State University, Warrensburg, Missouri.*

Show the following relation between an elliptic integral of the third kind with a modulus of a special form and elliptic integrals of the first and third kind with a simpler modulus.

$$\prod(\alpha^2, \frac{2\sqrt{l}}{1+l}) = \ \mathrm{sgn}\ \left([1-\alpha^2]\left[(1-\alpha^2)^2 - k'^2\right]\right)\frac{(1+k')\sqrt{(1-\alpha_1^2)(k_1^2-\alpha_1^2)}}{(k^2-\alpha^2)}\prod(\alpha_1^2, k_1)$$

$$+\frac{k^2(1+k_1)}{2(k^2-\alpha^2)}K(k_1)$$

where

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \ ,$$

$$\prod(\theta, \alpha^2, k) = \int_0^{\theta} \frac{d\phi}{(1-\alpha^2\sin^2\phi)\sqrt{1-k^2\sin^2\phi}} \ ,$$

$$\prod(\alpha^2, k) = \prod(\frac{\pi}{2}, \alpha^2, k) \ ,$$

and

$$k = \frac{2\sqrt{l}}{1+l} \ , \ k' = \sqrt{1-k^2} \ , \ k_1 = \frac{1-k'}{1+k'} \ ,$$

$$(*) \qquad \alpha^2 = \frac{(1+k')^2}{2}\left[k_1 + \alpha_1^2 - \sqrt{(1-\alpha_1^2)(k_1^2-\alpha_1^2)}\right] .$$

Note: $k_1 = l$ for $l \leq 1$ and $\frac{1}{l}$ for $l > 1$.

Solution by the proposer. We begin with two standard relations among elliptic integrals [1,2], i.e.,

$$(1) \qquad (1-\alpha^2)(k^2-\alpha^2)\prod(\alpha^2,k)+\alpha^2k'^2\prod\left(\frac{k^2-\alpha^2}{1-\alpha^2},k\right)=k^2(1-\alpha^2)K(k)$$

and

$$(2) \qquad \prod(\phi,\alpha_1^2,k_1)=(1+k')\frac{(k^2-\alpha^2)\prod(\theta,\alpha^2,k)+(\alpha_2^2-k^2)\prod(\theta,\alpha_2^2,k)}{\alpha_2^2-\alpha^2}\ ,$$

where

$$\sin\phi=\frac{(1+k')\sin\theta\cos\theta}{\sqrt{1-k^2\sin^2\theta}}$$

and

$$(3) \qquad \alpha_2^2=\frac{(1+k')^2}{2}\left[k_1+\alpha_1^2+\sqrt{(1-\alpha_1^2)(k_1^2-\alpha_1^2)}\right]\ .$$

In equation (2), we take $\theta=\frac{\pi}{2}$, which gives complete elliptic integrals on the right-hand side. It is found by comparison with the results of independently obtained special cases that $\phi=\pi$, rather than zero. Equation (2) becomes

$$(4) \qquad 2\prod(\alpha_1^2,k_1)=(1+k')\big[(k^2-\alpha^2)\prod(\alpha^2,k)+(\alpha_2^2-k^2)\prod(\alpha_2^2,k)\big]/(\alpha_2^2-\alpha^2)\ .$$

When k can be written in the form $k=2\sqrt{l}/(1+l)$, it can be shown directly that

$$\alpha_2^2=\frac{k^2-\alpha^2}{1-\alpha^2}\ ,\qquad \alpha_2^2-k^2=\frac{-\alpha^2k'^2}{1-\alpha^2}\ .$$

We can easily solve $(*)$ for α_1^2 in terms of α^2:

$$\alpha_1^2=\frac{\alpha^2}{\alpha^2-1}\left[\frac{\alpha^2}{(1+k')^2}-k_1\right]\ .$$

Equations (1) and (4) can each be solved for $\prod(\alpha_2^2, k)$ and the results equated, giving, after a little algebra,

$$\prod(\alpha^2, \frac{2\sqrt{l}}{1+l}) = \operatorname{sgn}\left([1-\alpha^2]\left[(1-\alpha^2)^2 - k'^2\right]\right) \frac{(1+k')\sqrt{(1-\alpha_1^2)(k_1^2-\alpha_1^2)}}{(k^2-\alpha^2)} \prod(\alpha_1^2, k_1)$$

$$+ \frac{k^2(1+k_1)}{2(k^2-\alpha^2)} K(k_1) \ .$$

In obtaining the last equation, use has also been made of the standard relation [1,2],

$$K\left(\frac{2\sqrt{l}}{1+l}\right) = (1+k_1)K(k_1) \ .$$

References

1. I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series, and Products,* 4th ed., Academic Press, New York, 1980.
2. P.F. Byrd and M.D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists,* 2nd ed., Springer-Verlag, New York, 1971.