A GENERALIZATION OF YOUNG'S THEOREM

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Abstract. The well known "SOR" method is obtained from a one-part splitting of the system matrix A, using one parameter ω.

M. Sisler introduced a new method by using one parameter for the lower triangular matrix L. Later he combined the above two methods to get a two parametric method [7], [8], and [9].

D. Young considered yet another two parametric method. The two parameters weight the diagonal of a positive-definite and consistently ordered 2-cyclic matrix [6]. Removing Young's hypothesis that both parameters are in the interval $(0, 1]$, we generalized his theorem.

1. Introduction. To find the solution vector x to the linear system $Ax = b$, where A is a sparse $n \times n$ matrix and b is a given n-vector of complex n-space, usually A is not easy to invert. Therefore, one seeks an easy-to-invert part of A, say A_0. Hence

$$A = A_0 - A_1 \tag{1.1}$$

or equivalently,

$$A = A_0(I - A_0^{-1}A_1) = A_0(I - B) \tag{1.2}$$

where $B = A_0^{-1}A_1$ is called the *iteration matrix.* Relation (1.1) is called an *additive splitting* which defines the $\{x_k\}$ for an arbitrary fixed x_0 via,

$$A_0x_{k+1} - A_1x_k = b \qquad k = 0, 1, 2, \dots$$

or equivalently

$$x_{k+1} = A_0^{-1}A_1x_k + A_0^{-1}b \qquad k = 0, 1, 2, \dots$$
$$x_{k+1} = Bx_k + A_0^{-1}b \qquad k = 0, 1, 2, \dots$$

Looking at relation (1.1), it is clear that if $\{x_k\}$ converges at all, it must converge to $x_{\text{sol}} = A^{-1}b$ (vector solution), where $Ax_{\text{sol}} = b$. Relation (1.2) shows that $\{x_k\}$ converges to $x_{\text{sol}} = A^{-1}b$ for each x_0 if and only if $\rho(B) < 1$, where $\rho(B)$ is the spectral radius of B [1,6]. Use relation (1.2) to measure the asymptotic convergence R_∞ of the sequence $\{x_k\}$ where R_∞ is defined by $R_\infty = -\log \rho(B)$ which carries information on how fast the sequence $\{x_k\}$ converges. In fact, $\frac{1}{R_\infty}$ asymptotically represents the number of iterations that suffice to produce one additional decimal place of accuracy in x_k's.

The above splitting is called *stationary* since there is no altering of parameter from iteration to iteration. It is called *one part splitting* since each x_{k+1} depends only on one previous vector x_k.

Examples of one-part stationary splitting are represented in the following important iteration methods.

<u>JACOBI</u>: Choose

$$A_0 = D \ , \quad A_1 = L + U \ .$$

Then

$$B_{\text{jacobi}} = B_j = D^{-1}(L + U)$$

where D is the diagonal part of A and $-L$, $-U$ are strictly lower and upper triangular parts of A, respectively.

<u>S.O.R.</u>: The Successive Overrelaxation (SOR) method was developed independently by Frankel [2] and Young [3], [4] in 1950. Choose

$$A_0 = \frac{1}{\omega}D - L \ , \quad A_1 = \left(\frac{1}{\omega} - 1\right)D + U \ .$$

Then

$$B = B_\omega = (D - \omega L)^{-1}((1-\omega)D + \omega U) \ . \tag{1.3}$$

<u>MSOR</u>: The Modified Successive Overrelaxation (MSOR) method was first considered by Devogelaere [5] in 1958. Here is how it works. Consider the matrix A in the following form

$$A = \begin{pmatrix} D_1 & M \\ N & D_2 \end{pmatrix}$$

where D_1 and D_2 are square non-singular matrices. Use ω for the "red" equations corresponding to D_1 and ω' for the "black" equations corresponding to D_2 then

$$A_0 = \begin{pmatrix} \frac{1}{\omega} D_1 & 0 \\ N & \frac{1}{\omega'} D_2 \end{pmatrix}$$

and

$$A_1 = A_0 - A = \begin{pmatrix} \left(\frac{1}{\omega} - 1\right) D_1 & -M \\ 0 & \left(\frac{1}{\omega'} - 1\right) D_2 \end{pmatrix}$$

Therefore, iteration matrix $B_{(\omega,\omega')}$ is defined by

$$B_{(\omega,\omega')} = A_0^{-1} A_1 \begin{pmatrix} (1-\omega) I_1 & \omega F \\ \omega'(1-\omega) G & \omega\omega' GF + (1-\omega') I_2 \end{pmatrix} \tag{1.4}$$

where $F = -D_1^{-1} M$ and $G = -D_2^{-1} N$. Young [6] has proved that if A is positive definite, then

$$\rho(B_{\omega_b}) < \overline{\rho}(B_{(\omega,\omega')})$$

where $\overline{\rho}(B_{(\omega,\omega')})$ is the virtual spectral radius of $B_{(\omega,\omega')}$. Young also showed that B_1 (Gauss-Seidel iteration matrix) converges faster than MSOR if A is positive definite, $0 < \omega \leq 1$ and $0 < \omega' \leq 1$.

In this paper a generalization of Young's theorem (A is positive definite, $0 < \omega \leq 1$ and $0 < \omega' \leq 1$) will be given (Theorem 2.2).

2. Generalized MSOR Method.

Lemma 2.1. Let

$$A = \begin{pmatrix} D_1 & M \\ N & D_2 \end{pmatrix}$$

where D_1 and D_2 are non-singular matrices. Let $\rho(B_j) < 1$ and assume all the eigenvalues of B_j are real. If $0 < \omega \leq 1$ or $0 < \omega' \leq 1$, then the eigenvalues of $B_{(\omega,\omega')}$ are real.

Proof. According to Young [6]

$$(\lambda + \omega - 1)(\lambda + \omega' - 1) = \lambda\omega\omega'\mu^2$$

or equivalently

$$\lambda^2 - (2 - \omega - \omega' + \omega\omega'\mu^2)\lambda + (\omega - 1)(\omega' - 1) = 0 \tag{2.1}$$

If $\omega = 1$ or $\omega' = 1$ by equation (2.1) it is clear that λ is real. Assume that $\omega \neq 1$, $\omega' \neq 1$ and $0 < \omega' < 1$. Let $\triangle$ be the discriminant of the quadratic equation (2.1), i.e.,

$$\begin{aligned} \triangle &= (2 - \omega - \omega' + \omega\omega'\mu^2)^2 - 4(\omega - 1)(\omega' - 1) \\ \triangle &= (1 - \omega'\mu^2)^2\omega^2 - 2(\omega' - 2\omega'\mu^2 + {\omega'}^2\mu^2)\omega + {\omega'}^2 \end{aligned} \tag{2.2}$$

The parabola (2.2) has no x-intercept since the discriminant $\triangle'$ of equation (2.2) is negative, because

$$\begin{aligned} \triangle' &= (\omega' - 2\omega'\mu^2 + {\omega'}^2\mu^2)^2 - {\omega'}^2(1 - \omega'\mu^2) \\ \triangle' &= 4{\omega'}^2\mu^2(\mu^2 - 1) + 4{\omega'}^3\mu^2(1 - \mu^2) \ . \end{aligned}$$

Hence

$$\triangle' = 4{\omega'}^2\mu^2(1 - \mu^2)(\omega' - 1) \ . \tag{2.3}$$

Now by assumption since $\omega' < 1$ and $\mu^2 < 1$, relation (2.3) is negative. Therefore, parabola (2.2) has no x-intercept. But it is known that $(1 - \omega'\mu^2)^2 > 0$, then $\triangle$ is always positive which implies that equation (2.1) has real roots. Lemma 2.1 is true for the case $0 < \omega \leq 1$, because we can arrange $\triangle$ as the following

$$\triangle = (1 - \omega\mu^2)^2{\omega'}^2 - 2(\omega - 2\omega\mu^2 + \omega^2\mu^2)\omega' + \omega^2 \ .$$

<u>Theorem 2.2</u>. Let

$$A = \begin{pmatrix} D_1 & M \\ N & D_2 \end{pmatrix}$$

where D_1 and D_2 are non-singular matrices. Assume that all the eigenvalues of B_j are real and $\mu_1 = \rho(B_j) < 1$. If $0 < \omega \leq 1$ or $0 < \omega' \leq 1$, then

$$\rho(B_{(\omega,\omega')}) \geq \rho(B_1) \ ,$$

except for the following special cases. Choose ω and ω' such that $\omega\omega' > 1$ and either

$$(a) \qquad \frac{1}{\omega} + \frac{1}{\omega'} < 2 \quad \text{and} \quad \frac{(1-\omega)(1-\omega')}{1-\omega\omega'} < \mu_1^2$$

or

$$(b) \qquad \mu_1^2 > \frac{(\omega + \omega' - 2) + \sqrt{M}}{2(1+\omega\omega')}$$

where $M = \omega^2 + {\omega'}^2 + \omega\omega'(-6 - 4\omega\omega' + 4\omega + 4\omega')$.

Proof.

(i) Suppose $0 < \omega \leq 1$ and $0 < \omega' \leq 1$ (Young's theorem [6]). (A new proof is given which is easier than Young's. Use this proof to extend Young's theorem). In relation (2.1)

$$\lambda^2 - (2 - \omega - \omega' + \omega\omega'\mu^2)\lambda + (\omega - 1)(\omega' - 1) = 0$$

By assumption $0 < \omega \leq 1$ and $0 < \omega' \leq 1$ therefore,

$$b(\mu) = 2 - \omega - \omega' + \omega\omega'\mu^2 > 0 \tag{2.4}$$

and

$$\lambda_i = \frac{b(\mu_i) \pm \sqrt{b^2(\mu_i) - 4(\omega-1)(\omega'-1)}}{2} \;. \tag{2.5}$$

Because all λ_i's are real, by Lemma 2.1 the spectral radius of $B_{(\omega,\omega')}$ is given by

$$\rho(B_{(\omega,\omega')}) = \frac{b(\mu_1) + \sqrt{b^2(\mu_1) - 4(\omega-1)(\omega'-1)}}{2} \;. \tag{2.6}$$

Thus,

$$\frac{b(\mu_1) + \sqrt{b^2(\mu_1) - 4(\omega-1)(\omega'-1)}}{2} > \mu_1^2$$

or equivalently

$$\sqrt{b^2(\mu_1) - 4(\omega - 1)(\omega' - 1)} > 2\mu_1^2 - b(\mu_1) \tag{2.7}$$

since $0 < \omega \leq 1$ and $0 < \omega' \leq 1$

$$\frac{1}{\omega} + \frac{1}{\omega'} > 1 \tag{2.8}$$

or

$$\omega + \omega' > \omega\omega'$$
$$\omega + \omega' - 2 > \omega\omega' - 2 > \mu_1^2(\omega\omega' - 2) \tag{2.9}$$

relation (2.9) holds because $\mu_1^2 < 1$. Therefore,

$$-\mu_1^2(\omega\omega' - 2) + \omega + \omega' - 2 > 0$$

or

$$2\mu_1^2 - (2 - \omega - \omega' + \mu_1^2\omega\omega') > 0 \ .$$

Since the right hand side of relation (2.7) is positive, one can square both sides of relation (2.7). Therefore,

$$b^2(\mu_1) - 4(\omega - 1)(\omega' - 1) > 4\mu_1^4 - 4\mu_1^2 b(\mu_1) + b^2(\mu_1)$$
$$(1 - \omega\omega')\mu_1^4 - (2 - \omega - \omega')\mu_1^2 + (\omega - 1)(\omega' - 1) < 0 \tag{2.10}$$
$$(\mu_1^2 - 1)((1 - \omega\omega')\mu_1^2 - (\omega - 1)(\omega' - 1)) < 0 \ . \tag{2.11}$$

In this case one can show that

$$\frac{(1 - \omega)(1 - \omega')}{1 - \omega\omega'} < 1 \tag{2.12}$$

holds since

$$\frac{1}{\omega} + \frac{1}{\omega'} > 2 \ .$$

One has also the following relation

$$\frac{(1 - \omega)(1 - \omega')}{1 - \omega\omega'} < \mu_1^2 \tag{2.13}$$

because clearly

$$\omega' - 1 + \mu_1^2 \omega' < \omega' - 1 + \mu_1^2$$
$$\omega(\omega' - 1 + \mu_1^2 \omega') < \omega' - 1 + \mu_1^2 \ .$$

Hence

$$\omega\omega' - \omega - \omega' + 1 < \mu_1^2 (1 - \omega\omega')$$

since $1 - \omega\omega' > 0$ which implies

$$\frac{\omega\omega' - \omega - \omega' + 1}{1 - \omega\omega'} < \mu_1^2 \ .$$

This shows that inequality (2.13) is true. Thus by inequalities (2.12) and (2.13)

$$\frac{(1-\omega)(1-\omega')}{1 - \omega\omega'} < \mu_1^2 < 1 \ .$$

This implies that inequality (2.12) always holds because $\mu_1^2 - 1 < 0$, which means in this case

$$\rho(B_{(\omega,\omega')}) > \rho(B_1) \ .$$

Of course if we choose (without loss of generality) $\omega > 1$ and $\omega' < 1$ such that $\omega\omega' < 1$ then obviously

$$\frac{(1-\omega)(1-\omega')}{1 - \omega\omega'} < 0 \ .$$

Hence inequalities (2.13) and (2.12) always hold.

(ii) Assume (without loss of generality) $0 < \omega' \leq 1$ and

$$0 < \omega < \frac{2 - \omega'}{1 - \omega' \mu_1^2}$$

such that $\omega\omega' > 1$. By this assumption $b(\mu_1) > 0$.

Claim.

$$\rho(B_{(\omega,\omega')}) > \rho(B_1)$$

if and only if inequality (2.11) holds.

Proof of Claim.

$$\frac{1}{\omega} + \frac{1}{\omega'} < 2$$

or

$$\omega\omega' - \omega - \omega' + 1 > 1 - \omega\omega' \ .$$

Hence

$$\frac{(1-\omega)(1-\omega')}{1-\omega\omega'} < 1$$

since $1 - \omega\omega' < 0$. By assumption

$$\frac{(1-\omega)(1-\omega')}{1-\omega\omega'} < \mu_1^2 < 1 \tag{2.14}$$

which implies that inequality (2.11) be always true. Note that if

$$\frac{1}{\omega} + \frac{1}{\omega'} > 2 \ ,$$

then $\mu_1^2 > 1$.

(iii) Assume (without loss of generality) $\omega' \le 1$ and

$$\omega \ge \frac{2-\omega'}{1-\omega'\mu_1^2}$$

then $b(\mu_1) < 0$. Hence,

$$\rho(B_{(\omega,\omega')}) = \frac{-b(\mu_1) + \sqrt{b^2(\mu_1) - 4(\omega-1)(\omega'-1)}}{2} \ .$$

Suppose that

$$\rho(B_{(\omega,\omega')}) < \rho(B_1)\ ,$$

then by the same argument in (i)

$$(1+\omega\omega')\mu_1^4 + (2-\omega-\omega')\mu_1^2 + (\omega-1)(\omega'-1) > 0\ . \tag{2.15}$$

Inequality (2.15) holds if and only if either

$$\frac{(3-2\sqrt{2})\omega' + 2(\sqrt{2}-1){\omega'}^2}{1+4\omega'-4{\omega'}^2} < \omega < \frac{(3+2\sqrt{2})\omega' - 2(\sqrt{2}+1){\omega'}^2}{1+4\omega'-4{\omega'}^2} \tag{2.16}$$

or

$$\omega > \frac{(3+2\sqrt{2})\omega' - 2(\sqrt{2}+1){\omega'}^2}{1+4\omega'-4{\omega'}^2} \tag{2.17}$$

$$\omega < \frac{(3-2\sqrt{2})\omega' + 2(\sqrt{2}-1){\omega'}^2}{1+4\omega'-4{\omega'}^2}\ . \tag{2.18}$$

Note that

$$\mu_1^2 > \frac{4{\omega'}^3 - 2(5-\sqrt{2}){\omega'}^2 + 2(2-\sqrt{2})\omega' + 2}{2(1+\sqrt{2}){\omega'}^3 - (3+2\sqrt{2}){\omega'}^2} \tag{2.19}$$

because its denominator is negative and its numerator is positive for $0 < \omega' \leq 1$, hence inequality (2.19) implies that

$$\frac{(3+2\sqrt{2})\omega' - 2(\sqrt{2}+1){\omega'}^2}{1+4\omega'-4{\omega'}^2} < \frac{2-\omega'}{1-\omega'\mu_1^2}\ . \tag{2.20}$$

Then inequality (2.16) and inequality (2.18) cannot hold since it contradicts

$$\omega > \frac{2-\omega'}{1-\omega'\mu_1^2}$$

so inequality (2.17) always holds. Now that inequality (2.17) holds, inequality (2.15) is true if and only if

$$\mu_1^2 < \frac{-(2-\omega-\omega') - \sqrt{M}}{2(1+\omega\omega')} \quad \text{or} \quad \mu_1^2 > \frac{-(2-\omega-\omega') + \sqrt{M}}{2(1+\omega\omega')} \ . \tag{2.21}$$

For $\omega > 1$,

$$(\omega - 1)(\omega' + \frac{1}{\omega}) > 0$$

multiplying by $(\omega' - 1)$,

$$((\omega' - 1)\omega')\omega^2 - (\omega' - 1)^2 - (\omega' - 1) < 0 \ .$$

This implies $-(2-\omega-\omega') - \sqrt{M} < 0$. Then the first inequality of (2.21) cannot hold. Hence, inequality (2.15) holds when $\omega' < 1$,

$$\omega' > \frac{2-\omega'}{1-\omega'^2}$$

and the second inequality of (2.21) holds (part b).

Examples.

(1) Suppose $\mu_1^2 = 0.5$, let $\omega' = 0.8$ and $\omega = 1.6$, then $\omega\omega' > 1$,

$$\frac{(1-\omega)(1-\omega')}{1-\omega\omega'} = 0.428571 < \mu_1^2$$

and

$$\frac{1}{\omega} + \frac{1}{\omega'} < 2 \ .$$

Hence, all the conditions of case (a) hold. It follows that the spectral radii

$$\rho(B_{(\omega,\omega')}) < \rho(B_1)$$

where

$$\rho(B_{(\omega,\omega')}) = 0.48$$

and $\rho(B_1) = 0.5$.

(2) Suppose $\mu_1^2 = 0.5$, let $\omega' = 0.7$ and $\omega = 1.9$, then $\omega\omega' > 1$

$$\frac{(1-\omega)(1-\omega')}{1-\omega\omega'} = 1.5 > \mu_1^2$$

and

$$\frac{1}{\omega} + \frac{1}{\omega'} < 2 \; .$$

It follows that the spectral radii

$$\rho(B_{(\omega,\omega')}) > \rho(B_1)$$

where

$$\rho(B_{(\omega,\omega')}) = 0.5737$$

and $\rho(B_1) = 0.5$.

(3) Suppose $\mu_1^2 = 0.7$, let $\omega' = 0.7$ and $\omega = 2.5$, then

$$\frac{-(2-\omega-\omega') + \sqrt{M}}{2(1+\omega\omega')} = 0.66778 < \mu_1^2 \; .$$

Hence, all the conditions of case (b) hold. It follows that the spectral radii

$$\rho(B_{(\omega,\omega')}) < \rho(B_1)$$

where

$$\rho(B_{(\omega,\omega')}) = 0.683$$

and $\rho(B_1) = 0.7$.

(4) Suppose $\mu_1^2 = 0.5$, let $\omega' = 0.7$ and $\omega = 2.1$, then

$$\frac{-(2-\omega-\omega') + \sqrt{M}}{2(1+\omega\omega')} = 0.56 > \mu_1^2 \; .$$

It follows that the spectral radii

$$\rho(B_{(\omega,\omega')}) > \rho(B_1)$$

where

$$\rho(B_{(\omega,\omega')}) = 0.61$$

and $\rho(B_1) = 0.5$.

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