ELEMENTARY APPLICATIONS OF COMPLEX NUMBERS

Russell Euler

Northwest Missouri State University

In 1777, Leonard Euler introduced the imaginary number i with the property $i^2 = -1$. Perhaps the use of the word "imaginary" is infelicitous, however, it connotes the distrust with which complex numbers are viewed by beginning students. The purpose of this paper is to show that in many cases the evaluation of real integrals can be facilitated by using complex numbers and, as a bonus, sometimes two (real) results can be obtained from one computation.

The content of this paper is elementary but could be beneficial to some who teach lower level mathematics courses. Formal proof is not the point of the paper. Computations will be done formally and no justification, other than noting that the results are correct, will be given. For the reader who can manipulate complex numbers, this nonrigorous approach can be used as a source of motivation to study complex analysis – where both meaning and justification can be given. It is assumed that the complex number i can be manipulated like a real number.

The exponential function in complex analysis is defined for all complex numbers $z = x + iy$ by

$$e^z = e^x(\cos y + i \sin y) \ . \tag{1}$$

This definition can be motivated by formally substituting iy for y in the Maclaurin expansion

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$$

and rearranging the terms to get

$$e^{iy} = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!} \ ,$$

$$= \cos y + i \sin y \ , \tag{2}$$

which agrees with (1) when $x = 0$. As a consequence, we can consistently define

$$\cos y = \frac{e^{iy} + e^{-iy}}{2} = \cosh iy \tag{3}$$

and

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i} = -i \sinh iy \ . \tag{4}$$

One of the fundamental formulas from integral calculus is

$$\int e^{zx} dx = \frac{1}{z} e^{zx} + c \ , \tag{5}$$

where c is arbitrary constant and z is a nonzero parameter. Throughout this paper it is assumed that (5) is valid for all nonzero complex parameters $z = a + bi$.

Example. We will evaluate $\int_0^\pi \sin^2 x dx$ by using complex numbers rather than using the standard approach of employing the half-angle formula $\sin^2 x = \frac{1-\cos 2x}{2}$. From (4) we get

$$\int_0^\pi \sin^2 x dx = \left(-\frac{1}{8} e^{2ix} + \frac{x}{2} - \frac{1}{8} e^{-2ix} \right) \Big|_0^\pi = \frac{\pi}{2} \ .$$

Similarly,

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \cos^4 x dx &= \frac{1}{16} \int_0^{\frac{\pi}{2}} (e^{ix} + e^{-ix})^4 dx \\ &= \frac{1}{16} \int_0^{\frac{\pi}{2}} (e^{4ix} + 4e^{2ix} + 6 + 4e^{-2ix} + e^{-4ix}) dx \\ &= \frac{3\pi}{16} \ . \end{aligned}$$

This result can be obtained using integration by parts or by using an appropriate reduction formula. The above procedure is noticeably easier.

The standard ploy taught in calculus to evaluate $\int e^{ax} \cos bx dx$ is to integrate by parts twice and thereby obtain

$$\int e^{ax} \cos bx dx = \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2} \int e^{ax} \cos bx dx + c \ ,$$

or equivalently,

$$\frac{a^2 + b^2}{b^2} \int e^{ax} \cos bx dx = \frac{1}{b^2} e^{ax} (b \sin bx + a \cos bx) + c \ .$$

Hence,

$$\int e^{ax} \cos bx dx = e^{ax} \frac{(b \sin bx + a \cos bx)}{a^2 + b^2} + c_1 \tag{6}$$

where c_1 is an arbitrary constant. A similar computation gives

$$\int e^{ax} \sin bx dx = e^{ax} \frac{(a \sin bx - b \cos bx)}{a^2 + b^2} + c_2 \ . \tag{7}$$

To see how complex numbers can be used to obtain (6) and (7) simultaneously, substitute $z = a + ib$ into (5) and use (1) to get

$$\int e^{ax} (\cos bx + i \sin bx) dx = \frac{a - bi}{a^2 + b^2} e^{ax} (\cos bx + i \sin bx) + c_1 + ic_2 \ . \tag{8}$$

Equating real and imaginary parts in (8) gives the desired results.

Example. Laplace transforms are useful in solving a special class of differential equations. The Laplace transform of a function f is denoted by $L[f(x)]$ and is defined by

$$L[f(x)] = \int_0^\infty e^{-sx} f(x)dx \ .$$

So, if $s > 0$, then

$$L[\sin x] = \int_0^\infty e^{-sx} \sin x dx = \lim_{t\to\infty} \left. \frac{e^{-sx}(-s\sin x - \cos x)}{s^2+1} \right|_0^t = \frac{1}{s^2+1} \ .$$

To obtain more general results consider $\int x^n e^{(a+bi)x} dx$. The method of integration by parts yields

$$\int x^n e^{(a+bi)x} dx = \frac{1}{a+bi}\left[x^n e^{(a+bi)x} - n\int x^{n-1} e^{(a+bi)x} dx\right] \ ,$$

and so,

$$\int x^n e^{ax}(\cos bx + i\sin bx)dx = \frac{a-bi}{a^2+b^2}\left[x^n e^{ax}(\cos bx + i\sin bx)\right.$$

$$\left. - n\int x^{n-1} e^{ax}(\cos bx + i \sin bx)dx\right] + c_1 + ic_2 \ . \tag{9}$$

Equating real and imaginary parts in (9) gives the two well-known identities

$$\int x^n e^{ax} \cos\, bx dx = \frac{x^n e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$- \frac{n}{a^2+b^2}\int x^{n-1} e^{ax}(a\cos bx + b\sin bx)dx + c_1 \ ,$$

and

$$\int x^n e^{ax} \sin\, bx dx = \frac{x^n e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$$

$$- \frac{n}{a^2+b^2} \int x^{n-1} e^{ax}(a \sin bx - b \cos bx) dx + c_2 \ .$$

<u>Example</u>. The reaction rate to a particular drug dosage at time x hours after administration is measured by $r(x) = x^2 e^{-x}$ (in appropriate units). The total reaction to the given drug dosage is

$$\int_0^\infty r(x) dx = \lim_{t \to \infty} \int_0^t x^2 e^{-x} dx \ .$$

Using the above result three times with $a = -1$, $b = 0$, and $n = 2$, 1, 0 gives

$$\int_0^\infty r(x) dx = 2 \ .$$

Amid all of these formal computations suppose we replace b with ib in (6). Then, for $a^2 \neq b^2$,

$$\int e^{ax} \cos ibx dx = \frac{e^{ax}(ib \sin ibx + a \cos ibx)}{a^2 - b^2} + c_1 \ . \tag{10}$$

Using (3) and (4) to simplify (10) yields

$$\int e^{ax} \cosh bx dx = \frac{e^{ax}(-b \sinh bx + a \cosh bx)}{a^2 - b^2} + c_1 \ .$$

Starting with (7) and performing the analogous steps shows that

$$\int e^{ax} \sinh bx dx = \frac{e^{ax}(a \sinh bx - b \cosh bx)}{a^2 - b^2} + c$$

if $a \neq \pm b$.

It is also possible to use complex analysis to evaluate (real) improper integrals. For instance, by employing residue theory it can be shown that

$$\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \frac{\sqrt{2\pi}}{4} \ ,$$

$$\int_{-\infty}^\infty \frac{\sin x}{x} dx = \pi \ ,$$

and

$$\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2} dx = \frac{\pi}{4b^3}(1+ab)e^{-ab} \ ,$$

for $a > 0$ and $b > 0$.

These integrals can be found in [1] and [2]. Although these integrals could possibly be evaluated by using techniques involving only real numbers, it is noticeably easier to obtain these results by using complex numbers.

References

1. R.V. Churchill, J.W. Brown and R.F. Verhey, *Complex Variables and Applications*, 3rd ed., McGraw Hill, 1976.
2. L.L. Pennisi, *Elements of Complex Variables*, 2nd ed., Holt, Rinehart and Winston, 1976.