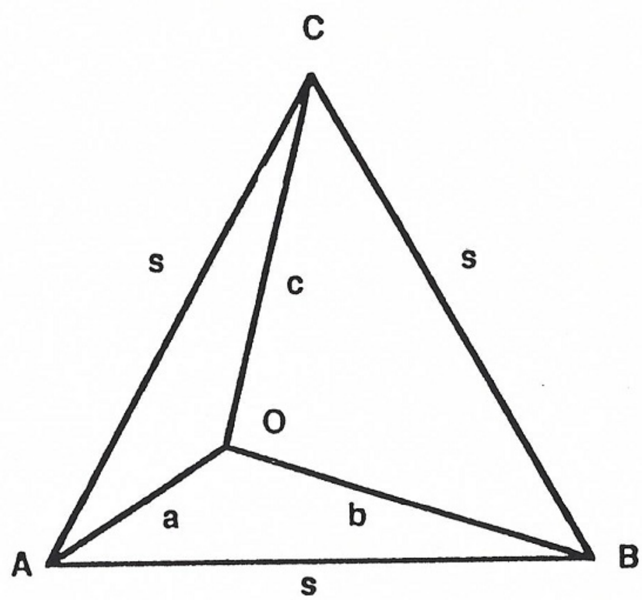
SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

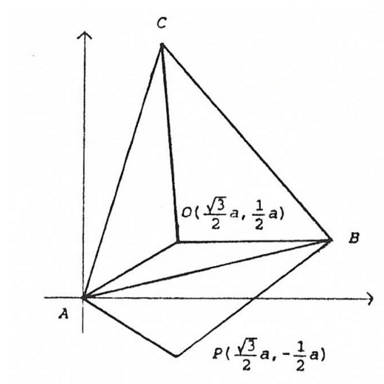
28. [1990, 141 and insert; 1991, 43; 1991, 156–157] *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Let ABC be an equilateral triangle with segment lengths as indicated in the diagram. Determine s as a function of a, b and c.



Solution by Robert L. Doucette, McNeese State University, Lake Charles, Louisiana.

Choose a rectangular coordinate system such that A coincides with the origin and AO lies in the first quadrant, forming a $30°$ angle with the positive x-axis. Let P be the reflection of the point O across the x-axis.



It isn't hard to see that $\angle PAB \cong \angle OAC$, and hence that $\triangle PAB \cong \triangle OAC$. Letting (x, y) be the coordinates of the point B, we have

$$(*) \qquad \left(x - \frac{\sqrt{3}}{2}a\right)^2 + \left(y - \frac{a}{2}\right)^2 = b^2 \ .$$

Also, since $PB \cong OC$,

$$\left(x - \frac{\sqrt{3}}{2}a\right)^2 + \left(y + \frac{a}{2}\right)^2 = c^2 \ .$$

We easily eliminate x, getting

$$y = \frac{c^2 - b^2}{2a} \ .$$

From $(*)$, and the position of the point B, we have

$$x = \frac{\sqrt{3}}{2}a + \left(b^2 - \left(\frac{c^2 - b^2}{2a} - \frac{a}{2}\right)^2\right)^{\frac{1}{2}} \ .$$

From $(*)$ we also have

$$s^2 = x^2 + y^2 = b^2 - a^2 + \sqrt{3}ax + ay \ .$$

Substituting the known values of x and y into this equation and simplifying,

$$s = \left(\frac{1}{2}\big(a^2 + b^2 + c^2\big) + \frac{\sqrt{3}}{2}\Big(2\big(a^2b^2 + a^2c^2 + b^2c^2\big) - \big(a^4 + b^4 + c^4\big) \Big)^{\frac{1}{2}} \right)^{\frac{1}{2}} \ .$$

29. [1991, 44] *Proposed by Jayanthi Ganapathy, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.*

Define a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ as follows:

$$a_1 = 5 \ ,$$

$$a_n = \sqrt{a_{n-1} + \sqrt{2a_{n-1}}} \ , \quad \text{for} \quad n \geq 2 \ .$$

Does $\{a_n\}$ converge and if so, to what?

Solution by Robert L. Doucette, McNeese State University, Lake Charles, Louisiana.

It is not hard to show that

$$a_n < a_{n-1} \ \text{ iff } \ 2 < a_{n-1}(a_{n-1} - 1)^2 \ .$$

Since decreasing sequences which are bounded below are convergent, to show that $\{a_n\}$ is convergent it suffices to show that $a_n > 2$, $n \geq 1$. This follows by induction:

$$a_1 > 2 \ \text{ and } \ a_{n-1} > 2 \ \text{ implies } \ a_n = \sqrt{a_{n-1} + \sqrt{2a_{n-1}}} > \sqrt{2 + \sqrt{2 \cdot 2}} = 2 \ .$$

Letting

$$a = \lim_{n \to \infty} a_n \ ,$$

we have from the given recurrence relation

$$a = \sqrt{a + \sqrt{2a}}$$

or

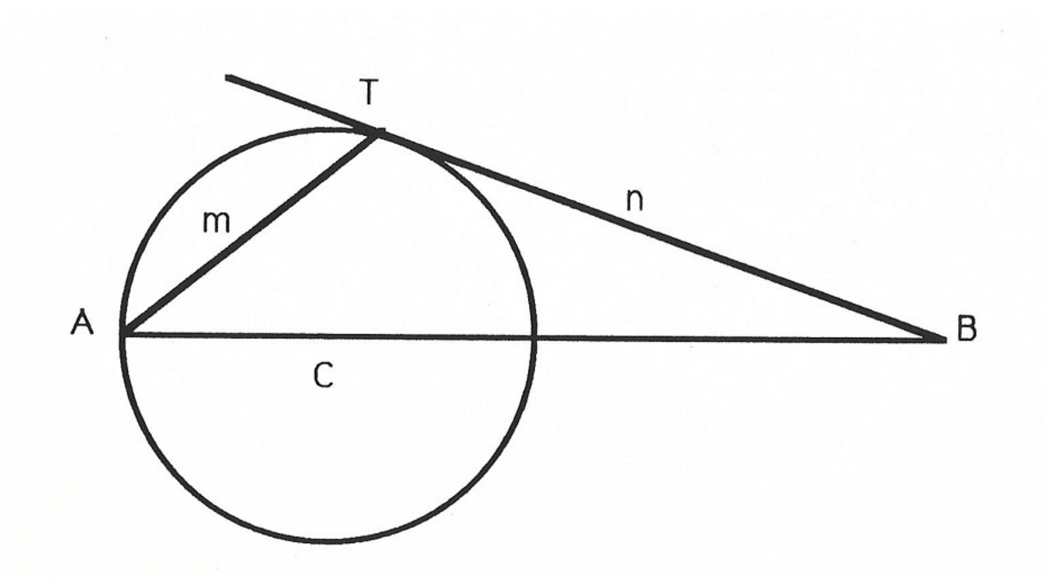
$$(a-2)(a^2+1)=0 \ .$$

Therefore, $\{a_n\}$ converges to 2. Note that the only assumption on a_1 is that it be greater than 2. For $0 < a_1 < 2$, a similar argument shows that $\{a_n\}$ is an increasing sequence which is bounded above. The same conclusion follows.

Also solved by Joseph E. Chance, University of Texas-Pan American; N. J. Kuenzi, University of Wisconsin-Oshkosh; Kandasamy Muthuvel, University of Wisconsin-Oshkosh; Leonard L. Palmer, Southeast Missouri State University; and the proposer.

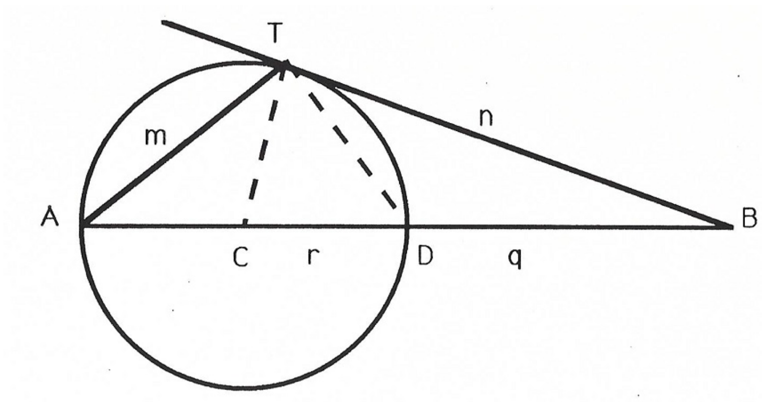
30. [1991, 44] *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

In the diagram below, C is the center of a circle of radius r, T is a point of tangency, $AT = m$ and $BT = n$. Determine r as a function of m and n.



Solution I by the proposer.

Let segment lengths be labeled as indicated in the diagram.



Using the law of cosines in triangle ABT gives

$$n^2 = m^2 + (2r+q)^2 - 2m(2r+q)\cos A\ . \tag{1}$$

Also, in the right triangles ADT and BCT,

$$\cos A = \frac{m}{2r}\ , \tag{2}$$

and

$$r^2 + n^2 = (r+q)^2\ . \tag{3}$$

Solving for q in equation (3) yields

$$q = -r + \sqrt{r^2+n^2}\ . \tag{4}$$

Substituting (2) and (4) into (1) and simplifying leads to

$$-2r^3 = (2r^2 - m^2)\sqrt{r^2+n^2}\ . \tag{5}$$

Squaring equation (5) and simplifying yields

$$4(m^2 - n^2)r^4 + (4m^2n^2 - m^4)r^2 - m^4n^2 = 0 \ ,$$

which has

$$r = \sqrt{\frac{m^2(m^2 - 4n^2) + m^3\sqrt{8n^2 + m^2}}{8(m^2 - n^2)}}$$

as the desired root.

Solution II by Robert L. Doucette, McNeese State University, Lake Charles, Louisiana.

Let D be the midpoint of segment AT and let α be the measure of $\angle CAT$. Then the measure of $\angle BCT$ is 2α. Hence

$$\tan\alpha = \frac{DC}{AD} = \frac{\sqrt{r^2 - \frac{m^2}{4}}}{\frac{m}{2}}$$

and

$$\tan 2\alpha = \frac{TB}{CT} = \frac{n}{r} \ .$$

From the identity

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} \ ,$$

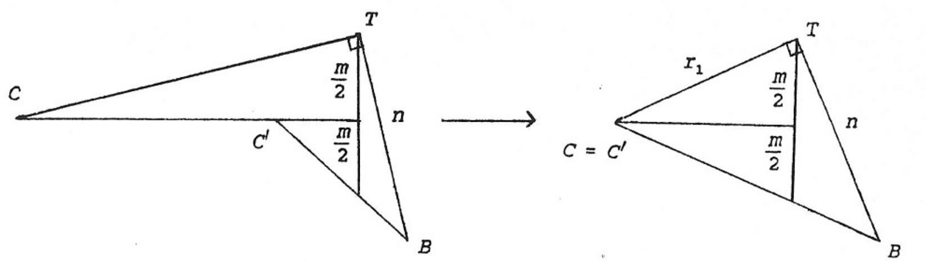
we conclude that

$$\frac{n}{r} = \frac{m\sqrt{4r^2 - m^2}}{m^2 - 2r^2} \ .$$

Hence r^2 is a root of

$$(*) \qquad 4(n^2 - m^2)x^2 + (m^4 - 4m^2n^2)x + m^4n^2 = 0 \ .$$

Suppose $n > m$. Referring to the figure below, we may rotate the right angle $\angle CTB$ about the point T in such a way that the points C and C' coincide.



As we did with r^2, we may show that r_1^2 is a root of $(*)$. Clearly, $r_1 > r$. With this we may use the quadratic formula to conclude that

$$(**) \qquad r^2 = \frac{m^2(4n^2 - m^2) - m^3\sqrt{m^2 + 8n^2}}{8(n^2 - m^2)} \, .$$

When $n < m$, one of the roots of $(*)$ is negative. Hence $(**)$ holds in this case as well. Lastly, when $n = m$, equation $(*)$ yields

$$r = \frac{m}{\sqrt{3}} \, .$$

31. [1991, 45] *Proposed by Troy L. Hicks, University of Missouri-Rolla, Rolla, Missouri.*

Put a topology t on $X = [0, 1]$ such that:

(a) (X, t) is an H-closed space, and

(b) (X, t) has a countable open cover such that no proper subfamily covers X.

(c) Every open cover has a finite subcollection whose closures cover.

<u>*Reference*</u>

1. S. Willard, *General Topology*, Addison-Wesley, 1970.

<u>Comments</u>. A Hausdorff space is H-closed if it is closed in every Hausdorff space in which it can be embedded. This generalizes a property of compact Hausdorff spaces. In [1], an example is given of an H-closed space X that is not compact. Also, it is noted that a Hausdorff space is H-closed iff every open cover has a finite subcollection whose closures cover.

Solution by the proposer.

For $x \neq 0$, we use the usual neighborhoods of x. The neighborhoods of 0 are supersets of sets of the form $[0, \epsilon)^*$ where $[0, \epsilon)^* = [0, \epsilon)$ minus points of the form $\frac{1}{n}$, n a positive integer, $0 < \epsilon \leq 1$. X is obviously a Hausdorff space.

$$\left(\frac{1}{2}, 1\right] \cup [0, 1)^* \cup \left(\bigcup_{n=2}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n-1}\right)\right)$$

is a countable open cover of X such that no subfamily covers X. The first set is the only set containing 1, the second set is the only set containing 0, the third set is the only set containing $\frac{1}{2}$, etc..

To show that X is H-closed we verify condition (c). Suppose

$$X = \bigcup_{\alpha \in A} G_\alpha$$

where each G_α is open. Then we have

$$\bigcup_{\alpha} G_\alpha = \left(\bigcup_{j} O_j\right) \cup \left(\bigcup_{k} [0, \epsilon_k)^*\right)$$

where each O_j is open in the usual topology for X. Let

$$a = \text{lub}_k \epsilon_k \ .$$

$$a \notin \bigcup_k [0, \epsilon_k)^*$$

implies $a \in O_j$ for some j, say $j = j'$. Now

$$a \in (c, d) \subset O_j \ ,$$

for some c and d. Choose k' such that

$$\epsilon_{k'} \in (c, d) \subset O_{j'} \ .$$

$$c < \epsilon_{k'} < a < d \ .$$

Note that

$$\bigcup_j O_j \supset [\epsilon_{k'}, 1] \ .$$

This is a covering of $[\epsilon_{k'}, 1]$ by usual open sets, so there exists $j_1, \cdots, j_n$ such that

$$\bigcup_{k=1}^{n} O_{j_k} \supset [\epsilon_{k'}, 1] \ .$$

Also,

$$\text{cl}[0, \epsilon_{k'})^* = [0, \epsilon_{k'}] \ .$$

Hence

$$\text{cl}[0, \epsilon_{k'})^* \cup \left(\bigcup_{k=1}^{n} O_{j_k} \right) \supset X$$

and X is H-closed.

32. [1991, 45] *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

Let

$$g(x) = \frac{45x + 1991}{x + 45} \ .$$

Evaluate

$$\lim_{k \to \infty} \underbrace{g(g(\cdots(g}_{k}(0))\cdots)) \ .$$

Solution I by Leonard L. Palmer, Southeast Missouri State University, Cape Girardeau, Missouri.

Let

$$f(x) = \frac{bx + a}{x + b} = b + \frac{a - b^2}{b + x} \ .$$

Then

$$f(f(x)) = b + \frac{a - b^2}{b + b + \frac{a-b^2}{x+b}} = b + \frac{a - b^2}{2b + \frac{a-b^2}{x+b}}$$

and

$$f(f(f(x))) = b + \frac{a - b^2}{2b + \frac{a-b^2}{2b+\frac{a-b^2}{2b+\frac{a-b^2}{b+x}}}} \ .$$

Let

$$\alpha = \lim_{n \to \infty} \underbrace{f(f(\cdots(f}_{n}(0))\cdots)) = b + \frac{a - b^2}{2b + \frac{a-b^2}{2b+\frac{a-b^2}{2b+\cdots}}} \ .$$

Then

$$\alpha + b = 2b + \frac{a - b^2}{b + \alpha}$$

and

$$(\alpha+b)^2 = 2b(\alpha+b) + a - b^2 \ .$$

So $\alpha^2 = a$, and $\alpha = \sqrt{a}$.

If $b = 45$ and $a = 1991$, then $g(x) = f(x)$ and $\alpha = \sqrt{1991}$.

Solution II by Robert L. Doucette, McNeese State University, Lake Charles, Louisiana.

Let

$$x_k = \underbrace{g(g(\cdots(g}_{k}(0))\cdots)) \ .$$

For $k \geq 1$, $x_{k+1} = g(x_k)$ and $x_k > 0$. For $x > 0$,

$$g'(x) = \frac{34}{(x+45)^2} < \frac{34}{45^2} \ .$$

Let $x^* = \sqrt{1991}$, the positive root of $g(x) = x$. By the Mean Value Theorem, there is a $y > 0$ such that

$$|x_{k+1} - x^*| = |g(x_k) - g(x^*)| \leq |g'(y)||x_k - x^*| \ .$$

Hence,

$$|x_{k+1} - x^*| \leq \frac{34}{45^2}|x_k - x^*| \ .$$

By induction we may show that, for $k \geq 0$,

$$|x_{k+1} - x^*| \leq \left(\frac{34}{45^2}\right)^k |x_1 - x^*| \ .$$

Since

$$\left(\frac{34}{45^2}\right)^k \to 0 \quad \text{as} \quad k \to \infty \ ,$$

$$\lim_{k\to\infty} x_k = x^* = \sqrt{1991} \ .$$

Solution III by Kandasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

Since

$$g(x) = 45 - \frac{34}{x+45} \ ,$$

$$g'(x) = \frac{34}{(x+45)^2} \ .$$

It is easy to see that

$$(*) \qquad \begin{array}{c} g \ \text{ is increasing on } \ [0,\infty) \\ \text{and} \\ g(x) < 45 \ \text{ for each } \ x \geq 0 \ . \end{array}$$

Let

$$u_k(0) = \underbrace{g(g(\cdots(g}_{k}(0))\cdots)) \ .$$

Since $0 < g(0) = u_1(0)$, by $(*)$,

$$g(0) = u_1(0) < g(g(0)) = u_2(0)$$

and hence by induction $\{u_k(0)\}$ is an increasing sequence of positive real numbers bounded above by 45. Now let

$$L = \lim_{k\to\infty} u_k(0) \ .$$

Then $0 < L \leq 45$. Since $u_{k+1}(0) = g(u_k(0))$ and g is continuous on $[0,\infty)$,

$$\begin{aligned} L &= \lim_{k\to\infty} u_{k+1}(0) \\ &= g\left(\lim_{k\to\infty} u_k(0)\right) \\ &= g(L) \\ &= \frac{45L+1991}{L+45} \ . \end{aligned}$$

This implies that $L = \sqrt{1991}$.

Remark. It is interesting to note that in the above problem if 45 is replaced by a positive number a and 1991 is replaced by a positive number b, then the value of the limit is $\sqrt{b}$.

Composite Solution IV by Joseph E. Chance, University of Texas-Pan American, Edinburg, Texas and Jayanthi Ganapathy, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

It is not too hard to show that if $x < \sqrt{1991}$, then

$$g(x) < \sqrt{1991} \tag{1}$$

and

$$x < g(x)\ . \tag{2}$$

The problem can be restated as follows:

Define $\{a_k\}$ by

$$a_1 = g(0) = \frac{1991}{45}$$

and

$$a_{k+1} = g(a_k) \ \text{ for } \ k = 1, 2, \dots\ .$$

Prove that

$$\lim_{k\to\infty} a_k$$

exists and find the limit.

Mathematical induction and (1) show that $a_k < \sqrt{1991}$ for $k \geq 1$. Also, (2) shows that $a_k < a_{k+1}$ for $k \geq 1$. Thus $\{a_k\}$ is bounded above and monotonically increasing, so it must have a limit. Let

$$a = \lim_{k\to\infty} a_k\ .$$

To find a, start with $a_{k+1} = g(a_k)$ and let $k \to \infty$. Thus, $a = g(a)$ or $a = \sqrt{1991}$. Therefore,

$$\lim_{k\to\infty} a_k = \sqrt{1991}\ .$$

Also solved by N. J. Kuenzi, University of Wisconsin-Oshkosh and the proposers.