ON THE AREA INSIDE A CIRCLE

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Every calculus book in print (that I know of) calculates

$$\lim_{x \to 0} \frac{\sin x}{x} \qquad (*)$$

in the same way. We estimate $\sin x \le x$ by the obvious comparison of a chord against an arc of a circle (Figure 1). The lower estimate is more interesting: we majorize the area of a certain sector by the area of a triangle that contains it (Figure 2). This leads to the estimate $\sin x/x \ge \cos x$. The Pinching Theorem then implies that $\sin x/x \to 1$.

Of course this limit plays a crucial role in the calculus because it allows us to compute the derivatives of the sine and cosine functions. A few hundred pages after the derivation of this limit and its consequences it is customary to illustrate the power of integral calculus and the Fundamental Theorem by calculating the area of a circle. We do this by integrating $\sqrt{1-x^2}$. This procedure is effected by performing a trigonometric substitution, which means that we must antidifferentiate the cosine function. But we learned how to differentiate (and antidifferentiate) the cosine function by means of the limit $(*)$. And calculating the limit $(*)$ entailed knowing the area of a sector, which in turn depends on knowing the area of a circle. Obviously this is all a bit circular.

So let us resort to the familiar method of exhaustion. Define π to be the quotient of the circumference of a circle by its diameter. We agree to measure the magnitude of an angle by the length of the chord it subtends on the unit circle. (Here length is defined in the usual fashion as the limit of lengths of piecewise linear approximations.) We inscribe in the unit circle a regular polygon with n sides (Figure 3). By breaking up the polygon into triangles (Figure 4), we find that

$$n \cdot \beta = 2\pi$$

and

$$n \cdot (2\alpha + \beta) = n \cdot \pi \ .$$

It results that
$$\alpha = \frac{\pi}{2} - \frac{\pi}{n} \ ,$$
the area of any one of the triangles is
$$\sin\alpha \cdot \cos\alpha \ ,$$
and (after substituting in α and simplifying) the area of the inscribed polygon is
$$n \cdot \sin\left(\frac{\pi}{n}\right) \cdot \cos\left(\frac{\pi}{n}\right) \ .$$

We calculate the area of the circle by determining the limit of this last expression as $n \to \infty$. And once again we need to know the limit $(*)$.

Thus what is needed is a method for calculating $(*)$ without reference to the area of a sector. Since $\sin x \le x$ is easy, we need more specifically to determine a suitable estimate from below. That is the purpose of this note.

Consider Figure 5. Since the marked angle is a right angle, we see that $\triangle ABC$ is similar to $\triangle OBA$. It follows that
$$\frac{d}{l} = \frac{\sin x}{1} \ .$$
Therefore
$$b < d = l \cdot \sin x \ .$$
But another look at the similar triangles $\triangle ABC$ and $\triangle OBA$ yields that $l = \tan x$. It follows that
$$b < \frac{\sin^2 x}{\cos x} \ .$$

But now observe, from [1] Figure 6 that
$$(**) \qquad x < b + \sin x$$

[1] In fact we should note that this is a special instance of an important analytic fact: if f is continuously differentiable and monotone on an interval $[\alpha, \beta]$ then the length of the graph of f on $[\alpha, \beta]$ does not exceed $|\beta - \alpha| + |f(\beta) - f(\alpha)|$. This in turn follows from an inspection of the way that we define arc length. More on this below.

which by the previous line is less than $\sin x \cdot (1 + \tan x)$. We conclude that

$$\frac{\sin x}{x} > \frac{1}{1 + \tan x} \ .$$

This is a suitable lower bound for applying the Pinching Theorem (since $\lim_{x \to 0} \sin x$ and $\lim_{x \to 0} \cos x$ are easy). We may therefore conclude that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \ .$$

A remark of an epistemological nature about the estimate $(**)$ is now in order. While this estimate is acceptable on heuristic grounds, it also follows from a careful look at the way we define length. Consider the piecewise linear approximation to the arc of length x exhibited in Figure 7. By the triangle inequality, the sum of the lengths of the linear segments does not exceed the sum of the rectilinear dotted segments. And their sum is precisely $b + \sin x$. Inequality $(**)$ follows.

We would be remiss not to point out that Archimedes's derivation of the area inside a circle was in no way circular. He used both inscribed and circumscribed polygons to obtain a recursion for the areas of said polygons. Thus he avoided the issues being addressed here.

The approach to the limit $\lim_{x \to 0} \sin x / x$ presented here was anticipated, at least in spirit, by the presentation in [1]. Our ideas were discovered independently. Since [1] is out of print, it is perhaps appropriate to revive these issues here.

Reference

1. R. P. Agnew, *Calculus*, McGraw-Hill, New York, 1962.

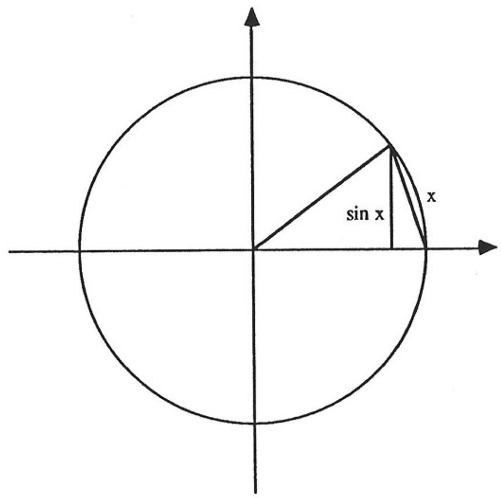

Figure 1.

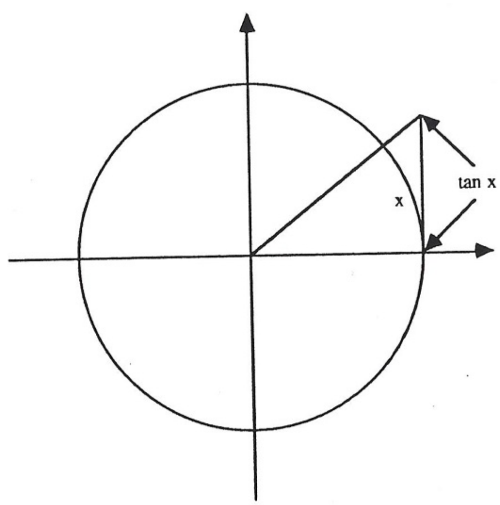

Figure 2.

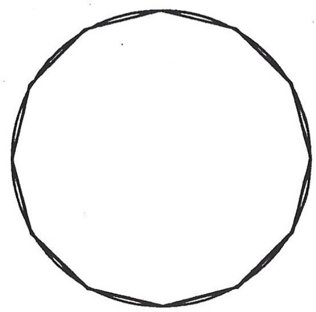

Figure 3.

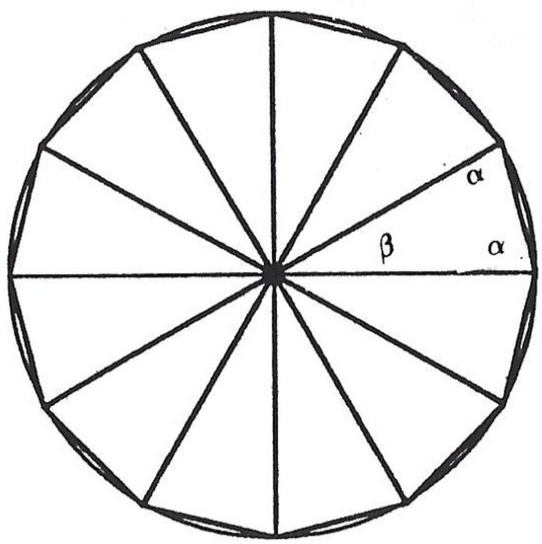

Figure 4.

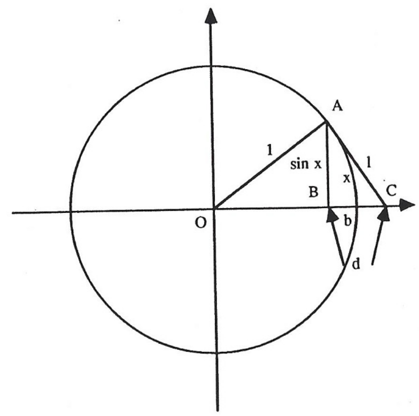

Figure 5.

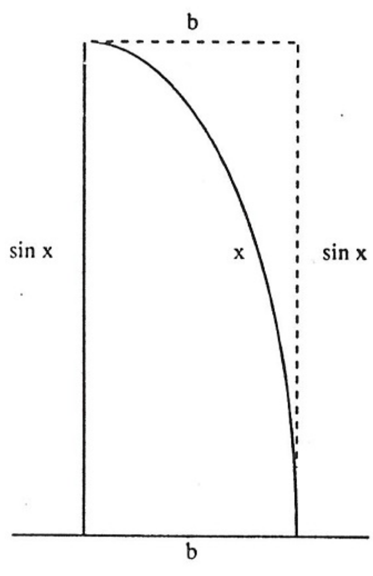

Figure 6.

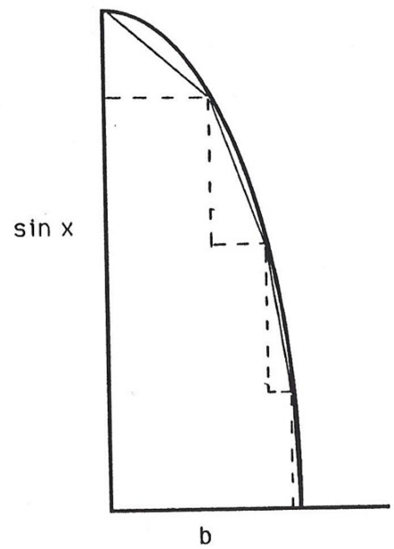

Figure 7.