## SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the editor.

**21**. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Find distinct positive integers, $a$, $b$, $c$, $d$ such that

$$a+b+c+d \quad + \quad abcd \quad = \quad ab+bc+ca+ad+bd+cd \quad + \quad abc+abd+acd+bcd \ .$$

*Solution by the proposer.*

My only solution is by computer search. The small solutions that I know of are:

$$a=2, b=4, c=16, d=70$$

and

$$a=2, b=4, c=22, d=32 \ .$$

*Comment by the editor.*

This problem is still wide open!

**22**. *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Without using Riemann sums prove that

$$\lim_{n\to\infty} n^{-3}\left(\sum_{k=1}^{n} k^{\frac{1}{2}}\right)^2 = \frac{4}{9} \ .$$

*Solution I by the proposer.*

From the Bernoulli Inequality we have

$$(1+k^{-1})^{\frac{3}{2}} > 1+\left(\frac{3}{2}\right)k^{-1} \ ,$$

$$(1-k^{-1})^{\frac{3}{2}} > 1-\left(\frac{3}{2}\right)k^{-1} \ .$$

Next, multiplying both of these two inequalities by $\frac{2}{3}k^{\frac{3}{2}}$, and then comparing them we have

$$\text{(1)} \qquad \frac{2}{3}\left(k^{\frac{3}{2}} - (k-1)^{\frac{3}{2}}\right) < k^{\frac{1}{2}} < \frac{2}{3}\left((k+1)^{\frac{3}{2}} - k^{\frac{3}{2}}\right) .$$

Now, if we write the double inequality (1) for $k = 1, 2, \ldots, n$, and then add the corresponding sides, we obtain

$$\text{(2)} \qquad \frac{2}{3}n^{\frac{3}{2}} < \sum_{k=1}^{n} k^{\frac{1}{2}} < \frac{2}{3}(n+1)^{\frac{3}{2}} .$$

Finally, squaring all three sides of the double inequality (2), and dividing the result by $n^3$, and then applying the Squeeze Theorem we achieve the desired equality.

*Solution II by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.*

It is known that if $r+1$ is positive, then

$$\lim_{n\to\infty} \frac{1^r + 2^r + \cdots + n^r}{n^{r+1}} = \frac{1}{r+1} .$$

[For a proof of this result (which does NOT use Riemann sums), see pp. 92–93 of Chrystal; Textbook of Algebra, Volume Two, Seventh Edition; Chelsea Publishing Company; New York, N.Y.; 1964.]

Thus

$$L = \lim_{n\to\infty} \frac{1^{\frac{1}{2}} + 2^{\frac{1}{2}} + \cdots + n^{\frac{1}{2}}}{n^{\frac{3}{2}}} = \frac{2}{3}$$

so

$$\lim_{n\to\infty} n^{-3}\left(\sum_{k=1}^{n} k^{\frac{1}{2}}\right)^2 = \lim_{n\to\infty}\left(\frac{1^{\frac{1}{2}} + 2^{\frac{1}{2}} + \cdots + n^{\frac{1}{2}}}{n^{\frac{3}{2}}}\right)^2 = L^2 = \frac{4}{9} .$$

**23**. *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

Let $n$ be a positive integer and $0 \leq i \leq 9$ be an integer. Let $g_i(n)$ denote the number of digits (base 10) in $n$ which are greater than or equal to $i$. Prove that

$$\sum_{n=1}^{\infty} \frac{g_2(5^n) + g_4(5^n) + g_6(5^n) + g_8(5^n)}{5^n} = \frac{5}{9} \ .$$

*Solution by the proposers.*

We will start with a couple lemmas. In what follows, $[\cdot]$ will denote the greatest integer function and

$$\{x\} = x - [x]$$

for any real number $x$.

Lemma 1. Let $k$ be a positive integer and $x$ be a real number. Then

$$[kx] - k[x] = [k\{x\}] \ .$$

Proof. Let $0 \leq i < k$ be an integer and suppose

$$\frac{i}{k} \leq \{x\} < \frac{i+1}{k} \ .$$

Then

$$[x] + \frac{i}{k} \leq x < [x] + \frac{i+1}{k} \ .$$

Therefore,

$$[kx] - k[x] = i = [k\{x\}] \ .$$

Lemma 2. Let $n$ and $k$ be positive integers and let $s(n)$ denote the digital sum (base 10) of $n$. Then

$$s(kn) = ks(n) - 9 \sum_{t \geq 1} \left[ k \left\{ \frac{n}{10^t} \right\} \right] \ .$$

Proof. It is well-known that

$$s(n) = n - 9\sum_{t\geq 1}\left[\frac{n}{10^t}\right] .$$

Therefore, using the above and Lemma 1,

$$\begin{aligned}
s(kn) &= kn - 9\sum_{t\geq 1}\left[\frac{kn}{10^t}\right] \\
&= kn - 9k\sum_{t\geq 1}\left[\frac{n}{10^t}\right] + 9k\sum_{t\geq 1}\left[\frac{n}{10^t}\right] - 9\sum_{t\geq 1}\left[\frac{kn}{10^t}\right] \\
&= ks(n) - 9\sum_{t\geq 1}\left(\left[k\frac{n}{10^t}\right] - k\left[\frac{n}{10^t}\right]\right) \\
&= ks(n) - 9\sum_{t\geq 1}\left[k\left\{\frac{n}{10^t}\right\}\right]
\end{aligned}$$

and we have Lemma 2.

Now applying Lemma 2 for powers of 5 and simplifying the sum, we have that

$$\begin{aligned}
s(5^{n+1}) &= 5s(5^n) - 9\sum_{t\geq 1}\left[5\left\{\frac{5^n}{10^t}\right\}\right] \\
&= 5s(5^n) - 9\big(g_2(5^n) + g_4(5^n) + g_6(5^n) + g_8(5^n)\big) .
\end{aligned}$$

Therefore,

$$\frac{g_2(5^n) + g_4(5^n) + g_6(5^n) + g_8(5^n)}{5^n} = \frac{1}{9}\left(\frac{s(5^n)}{5^{n-1}} - \frac{s(5^{n+1})}{5^n}\right) .$$

Summing both sides of this equality as $n$ goes from 1 to $k$ and observing the telescoping nature of the RHS, we have that

$$\sum_{n=1}^{k} \frac{g_2(5^n)+g_4(5^n)+g_6(5^n)+g_8(5^n)}{5^n} = \frac{5}{9} - \frac{s(5^{k+1})}{9 \cdot 5^k} \ .$$

Finally taking the limit of both sides of the equality as $k \to \infty$ and using the fact that $s(n) \le 9 \log n + 9$ (here $\log n$ denotes the base 10 logarithm of $n$), the result follows.

**24**. *Proposed by Ana Witt (student), Missouri Southern State College, Joplin, Missouri.*

If a linear transformation $K$ on an $n$-dimensional vector space, $V$, has $n+1$ eigenvectors such that any $n$ of them span $V$, what can we say about the linear transformation $K$?

*Composite solution by the proposer, Joe Flowers, Northeast Missouri State University, and the editors.*

We can say $K$ is a multiple of the identity.

Denote the $n+1$ eigenvectors by $v_1$, $v_2$, …, $v_{n+1}$. Then,

$$\begin{aligned} K(v_1) &= \tau_1 v_1 \\ K(v_2) &= \tau_2 v_2 \\ &\vdots \\ K(v_{n+1}) &= \tau_{n+1} v_{n+1} \ . \end{aligned}$$

<u>Claim 1</u>. $\tau_1 = \tau_2 = \cdots = \tau_{n+1}$.
Since $\{v_2, v_3, \ldots, v_{n+1}\}$ spans $V$, $v_1$ can be written in the form

$$v_1 = a_2 v_2 + a_3 v_3 + \cdots + a_{n+1} v_{n+1} \ ,$$

for some constants $a_2$, $a_3$, …, $a_{n+1}$. Furthermore, $a_2 \neq 0$, $a_3 \neq 0$, …, $a_{n+1} \neq 0$. To see this, suppose that $a_2 = 0$. Then

$$v_1 - a_3 v_3 - \cdots - a_{n+1} v_{n+1} = 0 \ ,$$

which contradicts the fact that $\{v_1, v_3, \ldots, v_{n+1}\}$ is linearly independent. Therefore, $a_2 \neq 0$. Similarly, $a_3 \neq 0$, …, $a_{n+1} \neq 0$.

Next,

$$\begin{aligned} K(v_1) &= K(a_2v_2 + a_3v_3 + \cdots + a_{n+1}v_{n+1}) \\ &= a_2K(v_2) + a_3K(v_3) + \cdots + a_{n+1}K(v_{n+1}) \\ &= a_2\tau_2v_2 + a_3\tau_3v_3 + \cdots + a_{n+1}\tau_{n+1}v_{n+1} \ . \end{aligned}$$

But also,

$$\begin{aligned} K(v_1) &= \tau_1v_1 \\ &= \tau_1(a_2v_2 + a_3v_3 + \cdots + a_{n+1}v_{n+1}) \\ &= a_2\tau_1v_2 + a_3\tau_1v_3 + \cdots + a_{n+1}\tau_1v_{n+1} \ . \end{aligned}$$

Thus

$$a_2\tau_2v_2 + a_3\tau_3v_3 + \cdots + a_{n+1}\tau_{n+1}v_{n+1} = a_2\tau_1v_2 + a_3\tau_1v_3 + \cdots + a_{n+1}\tau_1v_{n+1}$$

so

$$a_2(\tau_2 - \tau_1)v_2 + a_3(\tau_3 - \tau_1)v_3 + \cdots + a_{n+1}(\tau_{n+1} - \tau_1)v_{n+1} = 0 \ .$$

But $\{v_2, v_3, \ldots, v_{n+1}\}$ is linearly independent and $a_2 \neq 0$, $a_3 \neq 0$, $\ldots$, $a_{n+1} \neq 0$, so

$$\tau_1 = \tau_2 = \cdots = \tau_{n+1} \ .$$

Claim 2. $K(v) = \tau_1I(v)$ for all $v$ in $V$.

Let $v \in V$. Then

$$v = b_2v_2 + b_3v_3 + \cdots + b_{n+1}v_{n+1}$$

for some constants $b_2$, $b_3$, $\ldots$, $b_{n+1}$. Thus

$$\begin{aligned} K(v) &= K(b_2v_2 + b_3v_3 + \cdots + b_{n+1}v_{n+1}) \\ &= b_2K(v_2) + b_3K(v_3) + \cdots + b_{n+1}K(v_{n+1}) \\ &= b_2\tau_1v_2 + b_3\tau_1v_3 + \cdots + b_{n+1}\tau_1v_{n+1} \\ &= \tau_1(b_2v_2 + b_3v_3 + \cdots + b_{n+1}v_{n+1}) \\ &= \tau_1v \\ &= \tau_1I(v) \ . \end{aligned}$$

*Comment by the editor.*

This problem is essentially Problem A-6 on the 49th William Lowell Putnam Mathematical Competition (1988). For another solution to this problem, see L. F. Klosinski, G. L. Alexanderson, and Loren C. Larson, "The William Lowell Putnam Mathematical Competition," The American Mathematical Monthly, 96 (1989), 688–695.