SOME APPLICATIONS OF THE STONE-WEIERSTRASS THEOREM

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Abstract. In this paper, we will apply the Stone-Weierstrass Theorem to study some properties of the spaces of continuous functions on compact spaces. As a consequence, we will be able to construct a decreasing sequence of linear span dense subspaces in $C[0,1]$.

1. **Introduction and Definitions.** Let $C[a,b]$ denote the metric space of all continuous real valued functions on the closed bounded interval $[a,b]$ with the metric d defined by:

$$d(f,g) = ||f-g|| = \max |f(x) - g(x)|$$

for f, $g \in C[a,b]$ and $a \leq x \leq b$. Under this norm, $C[a,b]$ is a complete metric space. Similarly, we denote $C(M)$ to be the set of all continuous real valued functions on M, and we define:

$$||f|| = \max |f(x)|$$

for $f \in C(M)$, $x \in M$. If

$$d(f,g) = ||f-g||$$

for f, $g \in C(M)$, then d is a metric for $C(M)$. Let E be any set and let F be a family of complex valued functions on E. We say that F is an algebra if it is closed under the operations of addition, multiplication, and multiplication by constants. Furthermore, let G be a subset of F. We say that the family G separates points of E if given $x \neq y$ in E, there is an element $g \in G$ such that $g(x) \neq g(y)$.

In 1885, Weierstrass proved an important approximation theorem as follows:

Weierstrass Approximation Theorem [4]. Let f be any continuous function in $C[a,b]$. For any $\epsilon > 0$, there exists a polynomial P such that

$$|P(x) - f(x)| < \epsilon$$

for $a \leq x \leq b$. Equivalently, we say that the set P of all polynomials is dense in the complete metric space $C[a,b]$.

In 1937, M. H. Stone gave a remarkable generalization of Weierstrass' Theorem. This result applies to any compact space in place of $[a, b]$.

Stone – Weierstrass Theorem [4]. Suppose A is a self-adjoint algebra of complex continuous functions on a compact set K, A separates points on K, and A vanishes at no point of K. Then the uniform closure, B, of A consists of all complex continuous functions on K. In other words, A is dense in $C(K)$.

2. **Some Applications of the Stone-Weierstrass Theorem.**

Theorem 1. Let X be a compact Hausdorff space and let U be a subalgebra of $C(X)$ such that $f \in U$ then $\overline{f} \in U$ ($\overline{f}$ denotes the complex conjugate of the complex valued function f). Also, for any two points x, $y \in X$ with $x \neq y$, there is an element $g \in U$ with $g(x) \neq g(y)$. Then $\overline{U}$ (the uniform closure of U) either coincides with $C(X)$ or else $\overline{U}$ equals $\{f \in C(X) : f(x_0) = 0 \text{ for some } x_0 \in X\}$.

Proof. Case 1: For every $x \in X$, there is some function $f_x \in U$ with $f_x \neq 0$. We are going to show that $\overline{U} = C(X)$. Defining $g_x = f_x \cdot \overline{f}_x \in U$, we have that $g_x > 0$ since $f_x \neq 0$. By the continuity of g_x, we have $g_x > 0$ in some neighborhood V_x of x. It follows that there is an open cover $\{V_a : a \in X\}$ of X such that for each $a \in X$, there is a function $f_a \in U$ such that $f_a > 0$ in V_a. Since X is compact, there exists a finite subcover, say $V_{x_1}, V_{x_2}, \ldots, V_{x_N}$ where $g_{x_i} > 0$ in V_{x_i}. Now, defining

$$G = \sum_{i=1}^{N} g_{x_i} \ ,$$

we have that $G \in U$ and $G > 0$ on X. It follows that $1/G \in C(X)$. Let

$$B = \{c \cdot 1 + f : c \in C \ \text{ and } \ f \in U\} \ .$$

Then B is an algebra, satisfying all the conditions of the Stone-Weierstrass Theorem. It follows that there is a function $h \in B$ such that

$$|1/G - h| < \epsilon$$

where $\epsilon > 0$, or

$$(*) \qquad |1 - Gh| < \epsilon M$$

where $M = \sup\{G\}$ on X. Since $h \in B$, $Gh \in U$, and since ϵ is arbitrary, $(*)$ shows that $1 \in \overline{U}$. This shows that $\overline{U} = C(X)$.

Case 2: There is some $x_0 \in X$ with $f(x_0) = 0$ for all $f \in U$. We will show that

$$\overline{U} = \{f \in C(X) : f(x_0) = 0 \text{ for some } x_0 \in X\} .$$

Let $H \in C(X)$ and $H(x_0) = 0$. Using algebra B introduced in case 1, there exists a function $f \in U$, and a constant c such that $|H - (c + f)| < \epsilon/2$ on X, for any $\epsilon > 0$. Since $H(x_0) = 0 = f(x_0)$, we have $|c| < \epsilon/2$. It follows that $|H - f| < \epsilon$ on X. This shows that

$$\overline{U} = \{f \in C(X) : f(x_0) = 0 \text{ for some } x_0 \in X\} .$$

Theorem 2. For each integer $N \geq 1$, the set of functions

$$\{e^{nx} : n \geq N\}$$

has a linear span dense in $C[0, 1]$.

Proof. First, we put $y = e^x$. By the Weierstrass Approximation Theorem, polynomials in y are dense in $C[1, e]$. Moreover, given any $f \in C[1, e]$ and any $\epsilon > 0$, there is a finite sum

$$A_N y^N + A_{N+1} y^{N+1} + \cdots + A_{N+L} y^{N+L} = Q(y)$$

such that

$$|f(y) - Q(y)| < \epsilon$$

for all $y \in [1, e]$, and for all $N \geq 1$. To see this statement, we apply the Stone-Weierstrass Theorem to the function

$$y^{-N} = 1/y^N \in C[1, e] .$$

Also, for any $\delta > 0$, there is a polynomial $P_0(y)$ such that

$$|y^{-N} - P_0(y)| < e^{-2N} \cdot \delta$$

for $1 \leq y \leq e$. Thus there are finite sums $Q_0(y)$, $Q_1(y)$, $\ldots$, $Q_{N-1}(y)$, each of the special form

$$(*) \qquad A_N y^N + A_{N+1} y^{N+1} + \cdots + A_{N+L} y^{N+L}$$

such that

$$
\begin{aligned}
|1 - Q_0(y)| &< e^{-N} \cdot \delta \ \ ; \ \ 1 \le y \le e \ . \\
|y - Q_1(y)| &< e^{-N+1} \cdot \delta \ \ ; \ \ 1 \le y \le e \ . \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdot \\
|y^{N-1} - Q_{N-1}(y)| &< e^{-1} \cdot \delta \ \ ; \ \ 1 \le y \le e \ .
\end{aligned}
$$

By the Weierstrass Approximation Theorem, there is a polynomial $P(y)$ such that $|f(y) - P(y)| < \epsilon/2 \ \ ; \ \ 1 \le y \le e$. Now we can approximate terms of degree 0, 1, ... $N-1$ in $P(y)$ by the expressions of the form $(*)$. If $\delta > 0$ is small enough, we can replace $P(y)$ by an expression $Q(y)$ where

$$Q(y) = A_N y^N + A_{N+1} y^{N+1} + \cdots + A_{N+L} y^{N+L}$$

such that

$$(**) \qquad |f(y) - Q(y)| < \epsilon \ .$$

Secondly, given any $g \in C[0,1]$, if we define a function $f \in C[1,e]$ by $f(y) = g(x)$ where $y = e^x$, then $(**)$ becomes

$$|g(x) - Q(y)| < \epsilon \ .$$

This shows that $g(x)$ is approximated to within ϵ by a linear combination of

$$e^{Nx}, e^{(N+1)x}, \ldots, e^{(N+L)x} \ \text{ for } 0 \le x \le 1 \ .$$

Corollary. There is a decreasing sequence of linear span dense subspaces in $C[0,1]$.
Proof. Let

$$
\begin{aligned}
&L_1 = \mathrm{Span}\{e^x, e^{2x}, \ldots\} \\
&L_2 = \mathrm{Span}\{e^{2x}, e^{3x}, \ldots\} \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&L_N = \mathrm{Span}\{e^{Nx}, e^{(N+1)x}, \ldots\}
\end{aligned}
$$

Then $L_1 \supset L_2 \supset \cdots \supset L_N \supset \cdots$. By Theorem 2, L_i is dense in $C[0,1]$ for $i = 1, 2, 3, \ldots$.

Remark. Further applications of the Stone-Weierstrass Theorem can be found in [1], [2], [3], and [4].

References

1. J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, 1973.
2. R. Goldberg, *Methods of Real Analysis*, John Wiley and Sons, Inc., 1976.
3. G. Kelley, *General Topology*, D. Van Norstrand Company, Inc., 1955.
4. W. Rudin, *Principle of Mathematical Analysis*, McGraw-Hill, Inc., 1976.