HOMOTOPY EXTENSION THEOREM FOR A FIBER-PRESERVING PIECEWISE LINEAR μ-HOMOTOPY

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Abstract. In this paper, some nice extension properties of a PL R^∞-manifold will be investigated. As a consequence, a homotopy extension theorem for a fiber-preserving piecewise linear μ-homotopy will be proved.

1. **Introduction and Definitions.** R^∞-manifolds have been studied by Heisey and Sakai. In 1975, Heisey [2] proved that any R^∞-manifold is a countable direct limit of finite dimensional compact metric spaces. In 1981, Heisey [3] introduced the notion of a piecewise linear R^∞-structure and he defined the piecewise linear R^∞-manifold; then he proved that any separable paracompact piecewise linear R^∞-manifold may be regarded as a polyhedron in R^∞.

If $f : X \to Y$ is a function, we denote the image of f by $f(X)$. If $A \subset X$, then $f|A : A \to Y$ will denote the restriction of f to A. The symbol $\pi_x : X \times Y \to X$, will represent the projection onto X. Given a homotopy $f : X \times I \to Y$, and $t \in I$, the function $f_t : X \to Y$ is defined by $f_t(x) = f(x,t)$. If X is a space and $\{B_a | a \in A\}$ is a collection of subspaces of X, then X is said to have the weak topology generated by $\{B_a\}$ provided a set $U \subset X$ is open if and only if $U \cap B_a$ is open in B_a for all $a \in A$. Suppose for a sequence of spaces $\{X_n | n \geq 1\}$, X_n is a subspace of X_{n+1}, we define the direct limit of the sequence, denoted dirlim X_n, to be the set $\cup X_n$ with the weak topology generated by the collection $\{X_n\}$. In particular, $R^\infty = \text{dirlim } R^n$, as we consider the line $R^1 \subset R^2 \subset \cdots \subset R^n \subset \cdots$. Here, we think of R^∞ as

$$\{(x_i) : \text{ all but finitely many } x_i \text{ are } 0\} \ ,$$

and identify R^n with $R^n \times \{0\} \times \{0\} \times \cdots \subset R^\infty$. Let a be one-point set in R^n, and A, B be subsets of R^n, we say that aB is a cone with vertex a, and base B (or simply that aB is a cone) if each point not equal to a is expressed uniquely as $\lambda a + \mu b$ with $b \in b$, λ, $\mu \geq 0$ and $\lambda + \mu = 1$. A subset $P \subset R^n$ is called a $\underline{\text{polyhedron in the sense of Rourke} - \text{Sanderson}}$ [5] if for every point $x \in P$, there is a cone neighborhood xL, where L is compact. ∂L denotes the boundary of L. For this and other basic definitions and results from piecewise linear topology, see [5].

Following Heisey [3], we defined: A subset X of R^∞ is an R^∞-polyhedron if for each compact polyhedron C in R^∞, $C \cap X$ is a polyhedron in the usual sense of Rourke-Sanderson [5]. A map $f : X \to Y$ between two R^∞-polyhedra is an R^∞-piecewise linear (R^∞-PL) if for each compact polyhedron $C \subset X$ and any choice of n such that $f(C) \subset Y \cap R^n$, then $f|C : C \to Y \cap R^n$ is PL in the usual sense of Rourke-Sanderson [5]. For the convenience of notation, we will denote R^∞-piecewise linear by R^∞-PL. An R^∞-manifold X is a Hausdorff topological space such that for each $x \in X$, there exists a homeomorphism f_x mapping some open set containing x onto an open set in R^∞. A piecewise linear R^∞-atlas for a space M is a collection of pairs $\{(U_\alpha, \delta_\alpha)\}$ where $\{U_\alpha\}$ is an open cover of M by nonempty sets, $\delta_\alpha : U_\alpha \to \delta_\alpha(U_\alpha)$ is a homeomorphism onto an open subset of R^∞, and where, if $U_\alpha \cap U_\beta \neq \emptyset$, $\phi_\beta \phi_\alpha^{-1} : \delta_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is R^∞-PL. A piecewise linear R^∞-structure for M is a maximal PL R^∞-atlas for M; since any PL R^∞-atlas for the space M is contained in a unique maximal PL R^∞-atlas, a PL R^∞-atlas for M determines a PL R^∞-structure for M. A piecewise linear R^∞-manifold (PL R^∞-manifold) is a paracompact R^∞-manifold with a PL R^∞-structure which is defined by Heisey [3]. By Sakai [6], a PL R^∞-manifold can be regarded as an open set in R^∞. An R^∞-polyhedron in a PL R^∞-manifold is an R^∞-polyhedron in R^∞ as we regard it as an open subset in R^∞. Let $\triangle_n$ be an n-simplex, M and N be PL R^∞-manifolds, we define: a map $f : \triangle_n \times M \to \triangle_n \times N$ is $\mathrm{fiber-preserving}$ (fp) if $\pi\triangle_n \circ f = \pi\triangle_n$. By Sakai [7], $M = \mathrm{dirlim} M_n$ where each M_n is a compact polyhedral manifold in some $R^n \subset R^\infty$, that is, M_n is a compact polyhedron in R^n and a PL manifold; then we define a map $f : M \to N$ is R^∞-piecewise linear (R^∞-PL) if $f|M_n$ is piecewise linear in the sense of Rourke-Sanderson. An fp map f from an R^∞-polyhedron P in $\triangle_n \times M \to \triangle_n \times N$ is called R^∞-piecewise linear (R^∞-PL) if $\pi_N \circ f : P \to N$ is R^∞-PL. Two maps f and $g : M \to N$ are μ-close if given an open cover μ of N, and if for each $y \in M$, there is an open set $U \in \mu$ such that $\{f(y), g(y)\} \subset U$. Given any open cover μ of any space X, a homotopy $H : Y \times I \to X$ is an μ-homotopy if for each element $y \in Y$, the set $H(y \times I)$

is contained in some $U \in \mu$. A homotopy H is stationary on A or rel$A$ where $A \subset Y$ if we have $H(y \times I) = \{H(y, 0)\}$ for each $y \in A$. A homotopy $H : \triangle_n \times M \times I \to \triangle_n \times N$ is called R^∞-piecewise linear fiber − preserving homotopy if H_0 and H_1 are R^∞-PL fp maps where $H_0 : \triangle_n \times M \times 0 \to \triangle_n \times N$, $H_1 : \triangle_n \times M \times 1 \to \triangle_n \times N$ and H_t is an R^∞-PL for each $t \in [0, 1]$. Let μ, ν be families of subsets of X, $U \in \mu$, we define:

$$St(U, \nu) = \bigcup\{V | V \in \nu : U \cap V \neq \emptyset\}.$$

$$St(\mu, \nu) = \bigcup\{St(U, \nu) : U \in \mu\}.$$

Inductively, we define:

$$St^{n+1}(\mu, \nu) = St(St^n(\mu, \nu)) \;\text{ for } n > 1.$$

$\mu <^* \nu$ means that $St(\mu, \mu)$ is a refinement of ν. A closed subspace A of a space X is said to have the extension property in X with respect to a space Y, iff every map $f : A \to Y$ can be extended over X. The closed subspace A is said to have the neighborhood extension property in X with respect to Y, iff every map $f : A \to Y$ can be extended over some open subspace U of X which contains A. Here, the open subspace U of X may depend on the given map f.

Let C be any topological class of spaces, by an absolute extensor for the class C, (AE for the class C), we mean a space Y such that every closed subspace A of any space

X in the class C has the extension property in X with respect to Y. Similarly, by an absolute neighborhood extensor for the class C (ANE for the class C), we mean a space Y such that every closed subspace A of any space X in the class C has the neighborhood extension property in X with respect to Y.

2. **Extension of PL maps.**

Proposition 2.1. Let M, N be PL R^∞-manifolds. Let $A \subset B$ be two compact subpolyhedra of $\triangle_n \times N$. Let μ be an open cover of $\triangle_n \times M$ and $g : B \to \triangle_n \times M$ be a fp continuous map where $g|A$ is a fp R^∞-PL map, then f is μ-homotopic to g (fp) rel A.

Proof. By Sakai [6], we may regard M as an open set in R^∞. From compactness of $g(B)$, we have $g(B) \subset \triangle_n \times (M \cap R^p)$ for some $p > 0$. Consider a finite subcover of $g(B)$: $\{V_i \times U_i\}_{i=1,2,\dots,q}$. where V_i is a basis open set in $\triangle_n$, U_i is a basis open set in M and $V_i \times U_i$ is contained in some open set of μ. Since each U_i is a basis open set in R^∞, we may assume that

$$U_j = (-\epsilon_1^j, \epsilon_1^j) \times \cdots \times (-\epsilon_p^j, \epsilon_p^j) \times \cdots.$$

For each $j > 0$, let $\epsilon_{p+j} = \min\{\epsilon_{p+j}^1, \dots, \epsilon_{p+j}^q\}$. Observe that $M \cap R^p$ is a subset of $(M \cap R^p) \times (-\epsilon_{p+1}, \epsilon_{p+1}) \cdots$. Now let $m = 2(n + \dim B)$ and

$$\epsilon = \min\{\epsilon_{p+1}, \epsilon_{p+2}, \dots, \epsilon_{p+m}\}.$$

Consider $g : B \to \triangle_n \times (M \cap R^p) \subset \triangle_n \times (M \cap R^p) \times (-\epsilon, \epsilon)^m \subset \triangle_n \times M$. Let h be an

R^∞-PL embedding from B into $\partial(-\frac{\epsilon}{2}, \frac{\epsilon}{2})^m$. Let V be an open set containing A so that Cl $V \subset B$, then by Heisey [3] and Henderson [1], there is an R^∞-PL map $\Psi : B \to [0,1]$ so that $\Psi^{-1}(0) = A$ and $\Psi(B \backslash V) = 1$. First, we write $h(x) = (h_1(x), h_2(x), \ldots)$, then we define $f : B \to \triangle_n \times M$ by the following way: we write $f(x) = (f_1(x), f_2(x), \ldots)$ where $f_i(x) = \max\{-\Psi(x), \min\{\Psi(x), h_i(x)\}\}$ for $x \in B$; then f is the desired fp R^∞-PL map. Note that by the way we construct the embedding h, we have $\{f(x), g(x)\} \subset V_i \times U_i$ for some $i = 1, 2, 3, \ldots, q$. Therefore, by the convexity of $V_i \times U_i$, we can define a fp R^∞-PL homotopy $H(x,t) = tf(x) + (1-t)g(x)$, and H is the desired homotopy.

Proposition 2.2. Let $f : \triangle_n \times M \to \triangle_n \times N$ be a fp map, M and N are PL R^∞-manifolds, A a closed R^∞-subpolyhedron of $\triangle_N \times M$ and $f|A : A \to \triangle_N \times N$ is a fp R^∞-PL map, then there is a fp R^∞-PL map $g : \triangle_n \times M \to \triangle_n \times N$ such that $g|A = f|A$.

Proof. By Heisey [3], we regard M as a closed PL submanifold of R^∞. By Sakai [7], we write $M = \text{dirlim} M_n$ where each M_n is a compact polyhedral manifold in some $R^n \subset R^\infty$, and $M_n \subset M_{n+1}$. Similarly, write $N = \text{dirlim} N_n$ where each N_n is a compact polyhedral manifold in some $R^n \subset R^\infty$. It follows that $\triangle_n \times M = \text{dirlim}(\triangle_n \times M_n)$ and $\triangle_n \times N = \text{dirlim}(\triangle_n \times N_n)$. Let $A_i = A \cap (\triangle_n \times M_i)$ for $i = 1, 2, 3, \ldots$, then A_i is a compact R^∞-polyhedron and $A = \text{dirlim} A_i$. Now, we consider $f|\triangle_n \times M_1$ where $f|\triangle_n \times M_1 : \triangle_n \times M_1 \to \triangle_n \times N$ and $f|A_1$ is R^∞-PL, we may apply Proposition 2.1 to obtain a fp R^∞-PL map $g_1 : \triangle_n \times M_1 \to g(\triangle_n \times M_1) \subset \triangle_n \times N_j$ such that $g_1|A_1 = f|A_1$.

Now, extend g_1 to a fp R^∞-PL map $\tilde{g}_1$ as follows:

$$\tilde{g}_1(x) = f(x) \text{ if } x \in A_2 \ .$$

$$= g_1(x) \text{ if } x \in \triangle_n \times M_1 \ .$$

By Hu [4], there is an extension of $\tilde{g}_1$ to $\tilde{g}_2$ where $\tilde{g}_2 : \triangle_n \times M_2 \to \triangle_n \times N$ such that $\overline{g}_2|(\triangle_n \times M_1) \cup A_2 = \tilde{g}_1$. Applying Proposition 2.1 again, we may obtain a fp R^∞-PL map $g_2 : \triangle_n \times M_2 \to g_2(\triangle_n \times M_2) \subset \triangle_n \times N_k \subset \triangle_n \times N$ such that $g_2|(\triangle_n \times M_1) \cup A_2 = \tilde{g}_1$. Continuing this process, we obtain a family of fp R^∞-PL maps $\{g_n\}$ where $g_n : \triangle_n \times M_n \to \triangle_n \times N$ such that $g_n|(\triangle_n \times M_{n-1}) \cup A_n = \tilde{g}_{n-1}$. Therefore $\{g_n\}$ induces a fp R^∞-PL map g where $g : \triangle_n \times M \to \triangle_n \times N$ such that $g|A = f|A$.

Proposition 2.3. Let M, N be PL R^∞-manifolds, A a closed R^∞-polyhedron of M, μ an open cover of N. Let h be a homotopy of M to N such that $h|A \times I : A \times I \to N$, is a μ-homotopy; then there is a PL map $\Psi : M \to [0,1]$ such that $\Psi^{-1}(1) \supset A$ and $\{h(m \times [0, \Psi(m)])\} < \mu$.

Proof. Consider M as an open set in R^∞. Now, for each $t \in I$, $x \in A$, we have an open neighborhood V_t of x such that $h(V_t \times W_t) \subset U_x$. But $\{W_t : T \in I\}$ is an open cover of I and I is compact; there are finite numbers $t_1, \ldots, t_n \in I$ such that

$$I = \bigcup_{i=1}^{n} W_{t_i} \ .$$

Now, put

$$V_x = \bigcap_{i=1}^{n} V_{t_i} \ , \quad \text{and} \quad W = \bigcup_{a \in A} V_a \ ,$$

then $h|W \times I$ is a μ-homotopy. By Heisey [3], there is a PL map $\Psi : M \to I$ such that $\Psi|A = 1$ and $\Psi|M - W = 0$; then Ψ satisfies the conditions of the Proposition.

Theorem (Homotopy Extension Theorem). Let M and N be PL R^∞-manifolds and μ an open cover of $\triangle_n \times M$. Let A be a closed R^∞-subpolyhedron of $\triangle_n \times N$. Let $h : A \times I \to \triangle_n \times M$ be an R^∞-PL fp μ-homotopy such that $h_0 : A \to \triangle_n \times M$, has a fp R^∞-PL extension $f : \triangle_n \times N \to \triangle_n \times M$. Then h extends to $\tilde{h} : \triangle_n \times N \times I \to \triangle_n \times M$ which is an R^∞-PL fp μ-homotopy with $\tilde{h}_0 = f$. Moreover, if Γ is a compact subpolyhedron of $\triangle_n$ and if h is stationary on $A \cap (\Gamma \times N)$, then $\tilde{h}$ can be chosen to be stationary on $\Gamma \times N$.

Proof. Let $\overline{H} : ([\triangle_n \times N \times 0] \cup [A \cup \Gamma \times N]) \times I \to \triangle_n \times M$ be the map defined as follows:

$$\begin{aligned}\overline{H}(\lambda, x, t) = & h(\lambda, x, t) \text{ if } (\lambda, x) \in A \\ & f(\lambda, x) \text{ if } \lambda \in \Gamma\ . \\ & f(\lambda, x) \text{ if } t = 0\ .\end{aligned}$$

Observe that R^∞ is an AE for the countable direct limits of compact (metric) spaces, and M can be embedded in R^∞ as an open set. It is easy to see that M is ANE for the countable direct limits of compacta. Note that $\triangle_n \times N = \text{dirlim}\{\triangle_n \times N_g\}$ where each N_g is a PL finite dimensional compact submanifold of N_{g+1} and each $\triangle_n \times N_g$ is compact; it follows that M is ANE for the class $\triangle_n \times N$ where N is a PL R^∞-manifold. Now we consider

$\pi_M \circ \overline{H}|[A \cup \Gamma \times N] \times I : [A \cup \Gamma \times N] \times I \to M$. Since M is ANE for the direct limits of towers of compacta, there is an open set W in $\triangle_n \times N \times I$ containing $[A \cup \Gamma \times N] \times I$ such that $\pi_M \circ \overline{H}$ extends to W. Since I is compact, we may assume that $W = V \times I$ where $V \supset [A \cup \Gamma \times N]$ then $[A \cup \Gamma \times N]$ and $(\triangle_n \times N) \backslash V$ are disjoint closed subsets of the normal space $\triangle_n \times N$. By Urysohn's Lemma, there is a map $g : \triangle_n \times N \to I$ such that:

$$g(\lambda, x) = \begin{cases} 1 & \text{if } (\lambda, x) \in [A \cup \Gamma \times N] \\ 0 & \text{if } (\lambda, x) \in (\triangle_n \times N) \backslash V\ . \end{cases}$$

Now we extend $\pi_M \circ \overline{H}$ by $H(\lambda, x, t) = \pi_M \circ \overline{H}(\lambda, x, tg(\lambda, x))$. As a consequence of Proposition 2.2, there is an R^∞-PL map $G : \triangle_n \times N \times I \to M$ such that $G|[\triangle_n \times N \times 0] \cup [A \cup \Gamma \times N] \times I = H$. Now, we define: $\overline{h}(\lambda, x, t) = (\lambda, G(\lambda, x, t))$ for $(\lambda, x, t) \in \triangle_n \times N \times I$; then $\overline{h}$ is an R^∞-PL fp extension of h. Applying Proposition 2.3, there is a PL map $\phi : \triangle_n \times N \to I$ such that $\phi^{-1}(1) \supset [A \cup \Gamma \times N]$ and the family $\{\overline{h}(z \times [0, \phi(z)] : z \in \triangle_n \times N\} < \mu$. Now defining $\tilde{h}(\lambda, x, t) = \overline{h}(\lambda, x, t\phi(\lambda, x))$ for $(\lambda, x, t) \in \triangle_n \times N \times I$; then $\tilde{h}$ is the desired R^∞-PL fp μ-homotopy.

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