SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the editor.

7. *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Evaluate

$$L = \lim_{x \to 0} \left[\frac{\sin(\tan x) - \tan(\sin x)}{\sin^{-1}(\tan^{-1} x) - \tan^{-1}(\sin^{-1} x)} \right] .$$

Solution by Robert E. Kennedy and Curtis Cooper, Central Missouri State University, Warrensburg, Missouri.

It is well-known that as $x \to 0$

$$\begin{aligned} \sin x &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + O(x^9) \ , \\ \tan x &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + O(x^9) \ , \\ \sin^{-1} x &= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + O(x^9) \ , \\ \tan^{-1} x &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + O(x^9) \ . \end{aligned} \tag{1}$$

Using (1), as $x \to 0$

$$\begin{aligned} \sin(\tan x) &= \tan x - \frac{1}{6}\tan^3 x + \frac{1}{120}\tan^5 x \\ &\quad - \frac{1}{5040}\tan^7 x + O(\tan^9 x) \\ &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + O(x^9) \\ &\quad - \frac{1}{6}\big(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + O(x^9)\big)^3 \\ &\quad + \frac{1}{120}\big(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + O(x^9)\big)^5 \\ &\quad - \frac{1}{5040}\big(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + O(x^9)\big)^7 \\ &\quad + O(x^9) \end{aligned}$$

$$
\begin{aligned}
&= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + O(x^9) \\
&\quad - \frac{1}{6}\big(x^3 + x^5 + \frac{11}{15}x^7 + O(x^9)\big) \\
&\quad + \frac{1}{120}\big(x^5 + \frac{5}{3}x^7 + O(x^9)\big) \\
&\quad - \frac{1}{5040}\big(x^7 + O(x^9)\big) + O(x^9) \\
&= x + \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{55}{1008}x^7 + O(x^9)\ .
\end{aligned}
$$

Repeating this result and deriving three others similarly we have that as $x \to 0$

$$
\begin{aligned}
\sin(\tan x) &= x + \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{55}{1008}x^7 + O(x^9)\ , \\
\tan(\sin x) &= x + \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{107}{5040}x^7 + O(x^9)\ , \\
\sin^{-1}(\tan^{-1} x) &= x - \frac{1}{6}x^3 + \frac{13}{120}x^5 - \frac{341}{5040}x^7 + O(x^9)\ , \\
\tan^{-1}(\sin^{-1} x) &= x - \frac{1}{6}x^3 + \frac{13}{120}x^5 - \frac{173}{5040}x^7 + O(x^9)\ .
\end{aligned} \tag{2}
$$

Therefore, from (2) we have that as $x \to 0$

$$\sin(\tan x) - \tan(\sin x) = -\frac{1}{30}x^7 + O(x^9)$$

and

$$\sin^{-1}(\tan^{-1} x) - \tan^{-1}(\sin^{-1} x) = -\frac{1}{30}x^7 + O(x^9)\ .$$

Thus, as $x \to 0$

$$\frac{\sin(\tan x) - \tan(\sin x)}{\sin^{-1}(\tan^{-1} x) - \tan^{-1}(\sin^{-1} x)} = \frac{-\frac{1}{30}x^7 + O(x^9)}{-\frac{1}{30}x^7 + O(x^9)}\ .$$

Therefore,

$$L = \lim_{x\to 0}\left[\frac{\sin(\tan x) - \tan(\sin x)}{\sin^{-1}(\tan^{-1} x) - \tan^{-1}(\sin^{-1} x)}\right] = 1\ .$$

Also solved by the proposer.

8. *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

The Fibonacci numbers F_n satisfy $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n = 1, 2, 3, \ldots$. Find two solutions of $x^n = F_n x + F_{n-1}$ for all integers $n \geq 2$.

I. *Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.*

We shall show that $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the required solutions. (In fact when $n = 2$, the given equation becomes $x^2 - x - 1 = 0$ which has α and β as its only solutions.) Since $F_k = \frac{\alpha^k - \beta^k}{\alpha - \beta}$,

$$\begin{aligned}
\alpha^n - F_n\alpha &= \alpha^n - \frac{\alpha^n - \beta^n}{\alpha - \beta}\alpha \\
&= \frac{\alpha^{n+1} - \alpha^n\beta - \alpha^{n+1} + \alpha\beta^n}{\alpha - \beta} \\
&= \frac{(-\alpha\beta)(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta} \\
&= F_{n-1} \quad \text{(because } \alpha\beta = -1) \ .
\end{aligned}$$

Also

$$\begin{aligned}
\beta^n - F_n\beta &= \beta^n - \frac{\alpha^n - \beta^n}{\alpha - \beta}\beta \\
&= \frac{\alpha\beta^n - \beta^{n+1} - \alpha^n\beta + \beta^{n+1}}{\alpha - \beta} \\
&= \frac{(-\alpha\beta)(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta} \\
&= F_{n-1} \ .
\end{aligned}$$

This completes our solution.

II. *Composite solution by Charles J. Allard, Polo R-VII Schools, Polo, Missouri and Enis Alpakin (student), Central Missouri State University, Warrensburg, Missouri (independently).*

Suppose x is a solution of $x^n = F_n x + F_{n-1}$ for all integers

$n \geq 2$. Then for all integers $n \geq 1$,

$$x^{n+1} = F_{n+1}x + F_n \ ,$$
$$x \cdot x^n = F_{n+1}x + F_n \ ,$$
$$x(F_n x + F_{n-1}) = F_{n+1}x + F_n \ ,$$
$$F_n x^2 + (F_{n-1} - F_{n+1})x - F_n = 0 \ .$$

Now since $F_n \neq 0$ and $F_{n-1} - F_{n+1} = -F_n$ for $n \geq 1$,

$$x^2 - x - 1 = 0.$$

Therefore,

$$x = \frac{1 \pm \sqrt{5}}{2} \ .$$

III. *Composite solution by Joseph E. Chance, Pan American University, Edinburg, Texas; Alejandro Necochea, Pan American University, Edinburg, Texas; Leonard L. Palmer, Southeast Missouri State University, Cape Girardeau, Missouri; W. F. Wheatley III (student), Central Missouri State University, Warrensburg, Missouri; and the proposer (independently).*

We will show by induction on n that $x = \frac{1\pm\sqrt{5}}{2}$ are two solutions of $x^n = F_n x + F_{n-1}$ for all integers $n \geq 2$.
This statement is true for $n = 2$.
Assume the result is true for some $n \geq 2$. Then

$$\begin{aligned} x^{n+1} &= x^n \cdot x \\ &= (F_n x + F_{n-1}) \cdot x \\ &= F_n x^2 + F_{n-1}x \\ &= F_n(x+1) + F_{n-1}x \\ &= (F_n + F_{n-1})x + F_n \\ &= F_{n+1}x + F_n \ , \end{aligned}$$

so the result is true for $n + 1$.
Thus, by induction on n, $\frac{1\pm\sqrt{5}}{2}$ are solutions of $x^n = F_n x + F_{n-1}$ for all integers $n \geq 2$.