SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the editor.

3. *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

The Fibonacci numbers F_n satisfy $F_0 = 0$, $F_1 = 1$, and

$$F_{n+2} = F_{n+1} + F_n \text{ for } n = 0, 1, 2, \ldots \ .$$

Show that

$$\sum_{i=1}^{n} F_i^3 = \frac{1}{10} F_{3n+2} + \frac{3}{5}(-1)^{n-1} F_{n-1} + \frac{1}{2} \ .$$

Solution by Joe Flowers, Northeast Missouri State University, Kirksville, Missouri and Dale Woods, Central (Oklahoma) State University, Edmond, Oklahoma (jointly).

The proof is by induction on n and use of the well-known explicit formula $F_n = \frac{1}{\sqrt{5}}(a^n - b^n)$ where $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$. Note that $a \cdot b = -1$.

For $n = 1$,

$$\frac{1}{10} F_5 + \frac{3}{5} F_0 + \frac{1}{2} = \frac{5}{10} + 0 + \frac{1}{2} = 1,$$

as required. Now assume that

$$\sum_{i=1}^{k} F_i^3 = \frac{1}{10} F_{3k+2} + \frac{3}{5}(-1)^{k-1} F_{k-1} + \frac{1}{2} \text{ for some } k \geq 1 \ .$$

Then

$$\begin{aligned}
F_{k+1}^3 &= \frac{1}{5\sqrt{5}}(a^{k+1} - b^{k+1})^3 \\
&= \frac{1}{5\sqrt{5}}(a^{3k+3} - 3a^{2k+2}b^{k+1} + 3a^{k+1}b^{2k+2} - b^{3k+3}) \\
&= \frac{1}{5\sqrt{5}}(a^{3k+3} - b^{3k+3}) - \frac{1}{5\sqrt{5}}(3a^{k+1}b^{k+1})(a^{k+1} - b^{k+1}) \\
&= \frac{1}{5}F_{3k+3} - \frac{3}{5}(-1)^{k+1}F_{k+1},
\end{aligned}$$

and therefore

$$\begin{aligned}
&\sum_{i=1}^{k+1} F_i^3 = \sum_{i=1}^{k} F_i^3 + F_{k+1}^3 \\
&= \frac{1}{10}F_{3k+2} + \frac{3}{5}(-1)^{k-1}F_{k-1} + \frac{1}{2} + \frac{1}{5}F_{3k+3} - \frac{3}{5}(-1)^{k+1}F_{k+1} \\
&= \frac{1}{10}F_{3k+2} + \frac{1}{10}F_{3k+3} + \frac{1}{10}F_{3k+3} + \frac{3}{5}(-1)^k(F_{k+1} - F_{k-1}) + \frac{1}{2} \\
&= \frac{1}{10}F_{3k+4} + \frac{1}{10}F_{3k+3} + \frac{3}{5}(-1)^k F_k + \frac{1}{2} \\
&= \frac{1}{10}F_{3k+5} + \frac{3}{5}(-1)^k F_k + \frac{1}{2}\ .
\end{aligned}$$

Also solved by Russell Euler, Northwest Missouri State University, Maryville, Missouri and the proposers.

4. *Proposed by Curtis Cooper, Central Missouri State University, Warrensburg, Missouri.*

Let n be a positive integer and P be a permutation on $\{1, \ldots, n\}$. Which permutations result in non-contradictory lists of n statements where statement i in the list is

i. Statement $P(i)$ is false.

Solution by Charles J. Allard, Polo R-VII High School, Polo, Missouri.

The permutations leading to non-contradictory lists of n statements are the ones which are the products of disjoint cycles of even length.

To see this, let $(i_1, i_2, \ldots, i_m)$ be one of the cycles in the factorization of P. Then

$$\begin{aligned} i_1&.\ \text{Statement } i_2 \text{ is false.} \\ i_2&.\ \text{Statement } i_3 \text{ is false.} \\ &\vdots \\ i_m&.\ \text{Statement } i_1 \text{ is false.} \end{aligned}$$

Now all the statements cannot be false (since that would imply all the statements are true). Without loss of generality, suppose statement i_1 is true. Then statement i_2 is false so statement i_3 is true. Continuing in the above manner; if m is even, statement i_m is false; whereas if m is odd, statement i_m is true. Thus if m is even, statement i_1 is true; but if m is odd, statement i_1 is false, a contradiction.

Therefore, the only non-contradictory permutations are the ones which are the product of disjoint cycles of even length. It might be noted that if n is odd, none of the permutations are non-contradictory.

Also solved by the proposer.

5. *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

Show

$$\sum_{j=0}^{10}\left(2\cos\frac{2\pi j}{11}\right)^{11} = 22 \ .$$

Solution I. by Larry Eifler, University of Missouri - Kansas City, Kansas City, Missouri.

Let n be a positive integer and set $\omega = e^{\frac{2\pi i}{n}}$. Then

$$\begin{aligned} S_n &= \sum_{j=0}^{n-1}\left(2\cos\frac{2\pi}{n}j\right)^n \\ &= \sum_{j=0}^{n-1}\left(\omega^j + \omega^{-j}\right)^n \\ &= \sum_{j=0}^{n-1}\left[\sum_{k=0}^{n}\binom{n}{k}\omega^{jk}\omega^{-j(n-k)}\right] \\ &= \sum_{j=0}^{n-1}\left[\sum_{k=0}^{n}\binom{n}{k}\omega^{2jk}\right] \\ &= \sum_{k=0}^{n}\left[\binom{n}{k}\sum_{j=0}^{n-1}\omega^{2kj}\right] \ . \end{aligned}$$

Since

$$(\omega^{2k}-1)\sum_{j=0}^{n-1}\omega^{2kj} = \omega^{2kn} - 1 = 0 \ ,$$

we have

$$\sum_{j=0}^{n-1}\omega^{2kj} = \begin{cases} n, & \text{if } \omega^{2k} = 1; \\ 0, & \text{if } \omega^{2k} \neq 1 \ . \end{cases}$$

Thus,

$$S_n = \begin{cases} 2n, & \text{if } n \text{ is odd;} \\ 2n + n\binom{n}{\frac{n}{2}}, & \text{if } n \text{ is even} \ . \end{cases}$$

Solution II. by James Horner, Central Missouri State University, Warrensburg, Missouri.

Let

$$S(m,q) = \sum_{j=0}^{m-1} \left(2\cos\frac{q\pi j}{m}\right)^m$$

where q is an even integer and q and m are relatively prime. Also, let $x = \exp\left(\frac{q\pi i}{m}\right)$ and note that $x^p = 1$ if and only if p is an integer multiple of m.

Now,

$$\begin{aligned} S(m,q) &= \sum_{j=0}^{m-1} \left(x^j + x^{-j}\right)^m = \sum_{j=0}^{m-1} x^{-mj}\left(x^{2j}+1\right)^m \\ &= \sum_{j=0}^{m-1} \left(x^{2j}+1\right)^m = \sum_{j=0}^{m-1}\sum_{k=0}^{m} \binom{m}{k} x^{2jk} \\ &= \sum_{k=0}^{m} \binom{m}{k} \sum_{j=0}^{m-1} x^{2kj} \ . \end{aligned}$$

For $0 \le k \le m$, $x^{2k} = 1$ only when $k = 0$ and $k = m$. Then,

$$\sum_{j=0}^{m-1} x^{2kj} = \begin{cases} m, & \text{if } k = 0 \text{ or } k = m; \\ \frac{1-x^{2km}}{1-x^{2k}} = 0, & \text{if } k \neq 0 \text{ and } k \neq m \ . \end{cases}$$

Thus,

$$S(m,q) = \left[\binom{m}{0} + \binom{m}{m}\right] \cdot m = 2m \ .$$

Solution III. by James Horner, Central Missouri State University, Warrensburg, Missouri.

Let q and m be positive integers with qm even. With $x = \exp(\frac{q\pi i}{m})$,

$$\begin{aligned} S(m,q) &= \sum_{j=0}^{m-1}\left(2\cos\frac{q\pi j}{m}\right)^m = \sum_{j=0}^{m-1}\left(x^j + x^{-j}\right)^m \\ &= \sum_{j=0}^{m-1} x^{-mj}\left(x^{2j}+1\right)^m = \sum_{j=0}^{m-1} x^{-mj}\sum_{k=0}^{m}\binom{m}{k}x^{2jk} \\ &= \sum_{k=0}^{m}\binom{m}{k}\sum_{j=0}^{m-1} x^{(2k-m)j} \ . \end{aligned}$$

We note that $x^p = 1$ if and only if $\frac{qp}{m}$ is an even integer, and consider two cases.

When q is even, $x^{2k-m} = 1$ if and only if $\frac{qk}{m}$ is an integer. Also, $0 \le k \le m$. So, $x^{2k-m} = 1$ only when $k = 0$ or k is of the form $\frac{m}{p}$, where p is a common divisor of m and q.

Let $1 = p_1 < p_2 < p_3 < \cdots < p_n$ be the common divisors of m and q and let $k_0 = 0$ and $k_i = \frac{m}{p_i}$ for $1 \le i \le n$. Then,

$$\sum_{j=0}^{m-1} x^{(2k-m)j} = \begin{cases} m, & \text{if } k \in \{k_i : 0 \le i \le n\}; \\ \frac{1-x^{(2k-m)m}}{1-x^{2k-m}} = 0, & \text{otherwise} \ . \end{cases}$$

Thus, when q is even,

$$S(m,q) = m\sum_{i=0}^{n}\binom{m}{k_i} \ .$$

We note that if m and q are relatively prime (as, for example, $q = 2$ and $m = 11$), $S(m,q) = 2m$).

Suppose now that q is odd and m is even. In this case, $x^{2k-m} = 1$ if and only if $(\frac{2k}{m} - 1)$ is an even integer. So, we must have $\frac{2k}{m}$ an odd integer and we have $0 \leq k \leq m$. The only choice is $k = \frac{m}{2}$.

Thus, in this case, $S(m,q) = m\binom{m}{\frac{m}{2}}$.

Also solved by James Horner, Central Missouri State University, Warrensburg, Missouri (three solutions); Joseph Chance, Pan American University, Edinburg, Texas; Alejandro Necochea, Pan American University, Edinburg, Texas; Edward Wang, Wilfrid Laurier University, Waterloo, Ontario, Canada, and the proposers.

Wang notes that there is a well-known identity

$$\prod_{j=1}^{n} \sin \frac{\pi j}{n} = \frac{n}{2^{n-1}}$$

for all integers n. Thus from this and the problem we obtain the interesting identity

$$(*) \qquad \sum_{j=0}^{n-1} \left(\cos \frac{2\pi j}{n} \right)^n = \prod_{j=1}^{n} \sin \frac{\pi j}{n} \ .$$

for all odd integers n. Wang asks if there is a direct proof of (*).