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G-constellations and the maximal resolution of a quotient surface singularity

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ABSTRACT. For a finite subgroup G of $\mathrm{GL}(2, \mathbb{C})$, we consider the moduli space $\mathscr{M}_\theta$ of G-constellations. It depends on the stability parameter θ and if θ is generic it is a resolution of singularities of $\mathbb{C}^2/G$. In this paper, we show that a resolution Y of $\mathbb{C}^2/G$ is isomorphic to $\mathscr{M}_\theta$ for some generic θ if and only if Y is dominated by the maximal resolution under the assumption that G is abelian or small.

1. Introduction

The moduli spaces of G-constellations (on an affine space) are introduced in [CI04]. It is a generalization of the Hilbert scheme of G-orbits, which is denoted by G-Hilb. The moduli space depends on some stability parameter θ and the moduli space of θ-stable G-constellations is denoted by $\mathscr{M}_\theta$. If G is a subgroup of $\mathrm{SL}(n, \mathbb{C})$ acting on $\mathbb{C}^n$ and $n \leq 3$, then $\mathscr{M}_\theta$ is a crepant resolution of $\mathbb{C}^n/G$ for a generic stability parameter θ. The main result of [CI04] is that for a finite abelian subgroup $G \subset \mathrm{SL}(3, \mathbb{C})$ and for a projective crepant resolution $Y \to \mathbb{C}^3/G$, there is a generic stability parameter θ such that $Y \cong \mathscr{M}_\theta$. See [Kę14], [NdCS17], [Jun16] and [Jun18] for related results.

The purpose of this paper is to consider the case where G is a finite subgroup of $\mathrm{GL}(2, \mathbb{C})$. In this case, G-Hilb$(\mathbb{C}^2)$ is the minimal resolution of $\mathbb{C}^2/G$ by [Ish02] but $\mathscr{M}_\theta$ is a resolution which may not be minimal for generic θ (as we see in this paper). Then what is the condition for a resolution $Y \to \mathbb{C}^2$ to be isomorphic to some $\mathscr{M}_\theta$? One important observation is that there is a fully faithful functor (see Theorem 3)

$$D^b(\mathrm{coh}\, \mathscr{M}_\theta) \hookrightarrow D^b(\mathrm{coh}^G \mathbb{C}^2)$$

between the derived categories. According to the DK hypothesis [Kaw18], the inclusion of derived categories should be related with inequalities of canonical divisors. Then it is natural to ask if the following is true: Y is isomorphic to $\mathscr{M}_\theta$ for some θ if and only if Y is between the minimal and the maximal

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resolutions (see Conjecture 4), where the maximal resolution means the unique maximal one satisfying the inequality as in [KSB88]. The main result of this paper is the following. Recall that G is said to be small if it contains no pseudo reflection.

Theorem 1 (= Theorem 7). *Let* $G \subset \mathrm{GL}(2, \mathbb{C})$ *be a finite small subgroup and let* $X = \mathbb{C}^2/G$ *be the quotient singularity. Then a resolution of singularities* $Y \to X$ *is isomorphic to* $\mathcal{M}_\theta$ *for some* θ *if and only if* Y *is dominated by the maximal resolution.*

Conjecture 4 is a conjecture for general (not necessarily small) finite subgroups where the maximal resolution is defined for the pair of the quotient variety $\mathbb{C}^2/G$ and the associated boundary divisor. The "only if" part of the conjecture is proved in Proposition 1 by using the embedding of G into $\mathrm{SL}(3, \mathbb{C})$ and the fact that the moduli space of G-constellations for $G \subset \mathrm{SL}(3, \mathbb{C})$ is a crepant resolution of $\mathbb{C}^3/G$. We can show that the conjecture is true if G is abelian (Theorem 5) by using the result of [CI04]. The idea in the non-abelian case of Theorem 1 is to use iterated construction of moduli spaces as in [IINdC13] and reduce the problem to the abelian group case. Namely, let N be the cyclic group generated by $-I$, which is a normal subgroup of every non-abelian finite small subgroup. We consider G/N-constellations on the moduli space of N-constellations in §7. In order to do such iterated constructions, we define G-constellations on a general variety and consider their stability parameters in §6. A key to the proof of Theorem 1 is the description of the space of stability parameters for G/N-constellations on the moduli space of N-constellations, which is done in §8.1. The proof of Theorem 1 is completed in §8.2.

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2. G-constellations on $\mathbb{C}^n$

2.1. Definitions. Let $V = \mathbb{C}^n$ be an affine space and $G \subset \mathrm{GL}(V)$ a finite subgroup.

Definition 1. A *G-constellation* on V is a G-equivariant coherent sheaf E on V such that $H^0(E)$ is isomorphic to the regular representation of G as a $\mathbb{C}[G]$-module.

Let $R(G) = \bigoplus_{\rho \in \mathrm{Irr}(G)} \mathbb{Z}\rho$ be the representation ring of G, where $\mathrm{Irr}(G)$ denotes the set of irreducible representations of G. The parameter space of stability conditions of G-constellations is the $\mathbb{Q}$-vector space

$$\Theta = \{\theta \in \mathrm{Hom}_{\mathbb{Z}}(R(G), \mathbb{Q}) \mid \theta(\mathbb{C}[G]) = 0\},$$

where $\mathbb{C}[G]$ is regarded as the regular representation of G. The definition of the stability is based on the stability of quiver representations [Kin94]:

Definition 2. A G-constellation E is *θ-stable* (or *θ-semistable*) if every proper G-equivariant coherent subsheaf $0 \subsetneq F \subsetneq E$ satisfies $\theta(H^0(F)) > 0$ (or $\theta(H^0(F)) \geq 0$). Here the representation space $H^0(F)$ of G is regarded as an element of $R(G)$.

By virtue of King [Kin94], there is a fine moduli scheme $\mathcal{M}_\theta = \mathcal{M}_\theta(V)$ of θ-stable G-constellations on V.

Definition 3. We say that a parameter $\theta \in \Theta$ is *generic* if a θ-semistable G-constellation is always θ-stable.

There is a morphism $\tau : \mathcal{M}_\theta(V) \to V/G$ which sends a G-constellation to its support. It is a projective morphism if θ is generic (see [CI04, Proposition 2.2]).

2.2. Results of [CI04]**.** In this subsection, we recall results from [CI04]. Suppose $V = \mathbb{C}^3$ and let $G \subset \mathrm{SL}(V)$ be a finite abelian subgroup. For a generic parameter $\theta \in \Theta$, the morphism

$$\tau : \mathcal{M}_\theta \to \mathbb{C}^3/G$$

is a projective crepant resolution and we have a Fourier-Mukai transform

$$\Phi_\theta : D^b(\mathrm{coh}\, \mathcal{M}_\theta) \xrightarrow{\sim} D^b(\mathrm{coh}^G(\mathbb{C}^3)).$$

Here for a variety Y, coh Y denotes the category of coherent sheaves on Y and if Y is acted on by a finite group G, $\mathrm{coh}^G(Y)$ denotes the category of G-equivariant coherent sheaves on Y. The subset of Θ consisting of generic parameters is divided into chambers; the moduli space $\mathcal{M}_\theta$ and the equivalence Φ_θ depend only on the chamber to which θ belongs. Thus we write $\mathcal{M}_C$ and Φ_C instead of $\mathcal{M}_\theta$ and Φ_θ where C is the chamber that contains θ. We write

$$\varphi_C : K(\mathrm{coh}_0\, \mathcal{M}_C) \to K(\mathrm{coh}_0^G(\mathbb{C}^3))$$

for the induced isomorphism of the Grothendieck groups of the full subcategories $\mathrm{coh}_0\, \mathcal{M}_\theta$ and $\mathrm{coh}_0^G(\mathbb{C}^3)$ consisting of sheaves supported on the subsets $\tau^{-1}(0)$ and on $\{0\}$ respectively. Since $K(\mathrm{coh}_0^G(\mathbb{C}^3))$ has a basis consisting of skyscraper sheaves $\mathcal{O}_0 \otimes \rho$ with $\rho \in \mathrm{Irr}(G)$, it is naturally identified with $R(G)$.

The dual of φ_C is regarded as the map

$$\varphi_C^* : K(\mathrm{coh}^G(\mathbb{C}^3)) \to K(\mathrm{coh}\, \mathcal{M}_\theta)$$

between the Grothendieck groups of the categories of sheaves without restrictions on the supports. Then $K(\mathrm{coh}^G(\mathbb{C}^3))$ is identified with $\mathrm{Hom}(R(G), \mathbb{Z})$ and φ_C^* induces an isomorphism

$$\Theta \xrightarrow{\sim} F^1 K(\mathrm{coh}\, \mathcal{M}_\theta)_{\mathbb{Q}},$$

where $F^i K(\mathrm{coh}\, \mathcal{M}_\theta)$ is the subgroup consisting of the classes of objects whose supports are at least of codimension i.

On $\mathcal{M}_C$ there are tautological bundles $\mathcal{R}_\rho$ for irreducible representations ρ such that $\bigoplus_\rho \mathcal{R}_\rho \otimes_{\mathbb{C}} \rho$ has a structure of the universal G-constellation. For $\theta \in C$,

$$\mathcal{L}_C(\theta) := \bigotimes_\rho (\det \mathcal{R}_\rho)^{\otimes \theta(\rho)}$$

is the (fractional) ample line bundle on $\mathcal{M}_\theta$ obtained by the GIT construction. It coincides with the class

$$[\varphi_C^*(\theta)] \in F^1 K(\mathrm{coh}\, \mathcal{M}_C)_{\mathbb{Q}} / F^2 K(\mathrm{coh}\, \mathcal{M}_C)_{\mathbb{Q}} \cong \mathrm{Pic}(\mathcal{M}_C)_{\mathbb{Q}} \tag{2.1}$$

as in [CI04, §5.1]. Hence $[\varphi_C^*(\theta)] \in \mathrm{Amp}(\mathcal{M}_C)$ where $\mathrm{Amp}(\mathcal{M}_C)$ is the ample cone considered in $\mathrm{Pic}(\mathcal{M}_C)_{\mathbb{Q}}$. The main theorem of [CI04] and the argument in [CI04, §8] show the following:

Theorem 2 ([CI04]). *For any projective crepant resolution $Y \to \mathbb{C}^3/G$ and a class $l \in \mathrm{Amp}(Y)$, there exist a chamber C with $Y \cong \mathcal{M}_C$ and a parameter $\theta \in C$ satisfying $l = [\varphi_C^*(\theta)]$.*

Proof. The existence of a chamber C such that $Y \cong \mathcal{M}_C$ is [CI04, Theorem 1.1]. Moreover, [CI04, Proposition 8.2] ensures that we can find a chamber C and a parameter $\theta \in \bar{C}$ with $l = [\varphi_C^*(\theta)]$. Suppose $\theta \in \bar{C} \setminus C$. We have to see we can perturb θ in the fiber of $p \circ \varphi_C^*$ so that θ is in some chamber, where

$$p : F^1 K(\mathrm{coh}\, \mathcal{M}_C)_{\mathbb{Q}} \to \mathrm{Pic}(\mathcal{M}_C)_{\mathbb{Q}}$$

is the projection. Here recall that a wall of the chamber C is either the pre-image of a wall of the ample cone by $p \circ \varphi_C^*$ (type I or III) or does not contain a fiber of $p \circ \varphi_C^*$ (type 0); see [CI04, Theorem 5.9]. In our case, $p \circ \phi_C^*(\theta) = l$

is ample and therefore θ is on walls of type 0. Since the images of adjacent chambers in $F^1K(\mathrm{coh}\,\mathcal{M}_C)_{\mathbb{Q}}$ are related as in [CI04, (8.2) or (8.3)], we can perturb θ in the fiber of $p \circ \varphi_C^*$ and go out of walls.

2.3. G-constellations on $\mathbb{C}^2$. Let G be a finite subgroup of $\mathrm{GL}(2, \mathbb{C})$.

THEOREM 3. *If θ is generic, then the moduli space $\mathcal{M}_\theta$ is a resolution of singularities of $\mathbb{C}^2/G$. Moreover, the universal family of G-constellations defines a fully faithful functor*

$$\Phi_\theta : D^b(\mathrm{coh}\,\mathcal{M}_\theta) \to D^b(\mathrm{coh}^G\,\mathbb{C}^2).$$

PROOF. This is essentially Theorem 1.3 in the first arXiv version of [BKR01]. We have the inequality

$$\dim \mathcal{M}_\theta \times_{(\mathbb{C}^2/G)} \mathcal{M}_\theta \leq \dim \mathbb{C}^2$$

which is sharper than the assumption in [BKR01]. This allows us to apply the argument of [BKR01] (without using the triviality of the Serre functors) to show that Φ_θ is fully faithful and that $\mathcal{M}_\theta$ is smooth and connected (see [Ish02, Theorem 6.2]).

The problem we consider is to characterize the resolutions Y such that $Y \cong \mathcal{M}_\theta$ for some generic θ.

3. The maximal resolution

Let G be a finite subgroup of $\mathrm{GL}(2, \mathbb{C})$, which is not necessarily small, i.e., the action may not be free on $\mathbb{C}^2 \setminus \{0\}$. Then the quotient variety $X = \mathbb{C}^2/G$ is equipped with a boundary divisor B determined by the equality $\pi^*(K_X + B) = K_{\mathbb{C}^2}$. More precisely, B is expressed as

$$B = \sum_j \frac{m_j - 1}{m_j} B_j,$$

where $B_j \subset X$ is the image of a one-dimensional linear subspace whose pointwise stabilizer subgroup $G_j \subset G$ is cyclic of order m_j. Note that G is small if and only if $B = 0$. Let $\tau : Y \to X$ be a resolution of singularities and write

$$K_Y + \tau_*^{-1} B \equiv \tau^*(K_X + B) + \sum_i a_i E_i, \tag{3.1}$$

where E_i are the exceptional divisors and $a_i \in \mathbb{Q}$. Recall that (X, B) is a KLT pair ([KM98, Proposition 5.20]), which implies $a_i > -1$ for all i. Then among the resolutions Y which satisfy $a_i \le 0$ for all i, there is a unique maximal one, as in [KSB88] (see also [Kaw18, Theorem 17]). It is called the maximal resolution of (X, B) and we denote it by $Y_{\max}$.

Notice that the system of inequalities $a_i \le 0$ is an inequality between canonical divisors. According to the DK-hypothesis [Kaw18], the inequality should correspond to the embedding of derived categories in Theorem 3 with $Y = \mathscr{M}_\theta$. Thus we make the following conjecture:

Conjecture 4. *Let $G \subset \mathrm{GL}(2, \mathbf{C})$ be a finite subgroup and consider the quotient $X = \mathbf{C}^2/G$ with the boundary divisor B. For any resolution of singularities $Y \to X$, there is a generic $\theta \in \Theta$ with $Y \cong \mathscr{M}_\theta$ if and only if there is a morphism $Y_{\max} \to Y$ over X. Here $Y_{\max}$ is the maximal resolution of (X, B).*

4. "Only if" part

In this section, we show the "only if" part of Conjecture 4. Embed $\mathrm{GL}(2, \mathbf{C})$ into $\mathrm{SL}(3, \mathbf{C})$ by sending a matrix $A \in \mathrm{GL}(2, \mathbf{C})$ to $\begin{pmatrix} A & 0 \\ 0 & \det(A)^{-1} \end{pmatrix}$. Then for $\theta \in \Theta$, we can consider the moduli space $\mathscr{M}_\theta(\mathbf{C}^3)$ of θ-stable G-constellations on $\mathbf{C}^3$ with respect to the action of G on $\mathbf{C}^3$.

Lemma 1. *For any $\theta \in \Theta$, there is a closed embedding $\mathscr{M}_\theta \hookrightarrow \mathscr{M}_\theta(\mathbf{C}^3)$ which fits into the commutative diagram*

$$\begin{array}{ccc} \mathscr{M}_\theta & \hookrightarrow & \mathscr{M}_\theta(\mathbf{C}^3) \\ \downarrow & & \downarrow \\ \mathbf{C}^2/G & \hookrightarrow & \mathbf{C}^3/G. \end{array}$$

Moreover, if θ is generic for G-constellations on $\mathbf{C}^3$, then the vertical arrows are projective and hence are resolutions of singularities.

Proof. Recall that the universal family of G-constellations on $\mathbf{C}^3$ is given by the tautological bundles $\{\mathscr{R}_\rho\}_{\rho \in \operatorname{Irr} G}$ and the G-equivariant morphism

$$\bigoplus_\rho \mathscr{R}_\rho \otimes_{\mathbf{C}} \rho \to \mathbf{C}^3 \otimes \left(\bigoplus_\rho \mathscr{R}_\rho \otimes_{\mathbf{C}} \rho \right). \tag{4.1}$$

If ρ_{nat} denotes the representation given by $G \subset \mathrm{GL}(2, \mathbf{C})$, then $\mathbf{C}^3$ above is $\rho_{\mathrm{nat}} \oplus \det \rho_{\mathrm{nat}}^*$. Taking the third coordinate of $\mathbf{C}^3$ in (4.1) we obtain a morphism

$$z_\rho : \mathscr{R}_\rho \to \mathscr{R}_{\rho \otimes \det \rho_{\mathrm{nat}}}$$

for each ρ. It is straightforward that the scheme theoretic intersection of the zero loci of z_ρ's is isomorphic to $\mathcal{M}_\theta$. Hence $\mathcal{M}_\theta$ is a closed subscheme of $\mathcal{M}_\theta(\mathbf{C}^3)$. Moreover, we can see that the composite $\mathcal{M}_\theta \hookrightarrow \mathcal{M}_\theta(\mathbf{C}^3) \to \mathbf{C}^3/G$ factors through $\mathbf{C}^2/G$. If θ is generic for G-constellations on $\mathbf{C}^3$, then it is also generic for G-constellations on $\mathbf{C}^2$, from which the projectivities of the vertical arrows follow.

Now let us prove the "only if" part.

Proposition 1. *If θ is generic, then there is a morphism $Y_{\max} \to \mathcal{M}_\theta$ over X.*

Proof. Putting $Y = \mathcal{M}_\theta$, we show that $a_i \leq 0$ for all i in (3.1). Embed G into $\mathrm{SL}(3, \mathbf{C})$ and consider $U := \mathcal{M}_\theta(\mathbf{C}^3)$, the moduli space of θ-stable G-constellations on $\mathbf{C}^3$. Here, we may assume that θ is generic for G-constellations on $\mathbf{C}^3$ by slightly perturbing θ if necessary. Then U is a crepant resolution of $\mathbf{C}^3/G$ containing Y by Lemma 1 and therefore we have

$$K_Y \cong \mathcal{O}_U(Y)|_Y. \tag{4.2}$$

Let z be the coordinate function of $\mathbf{C}^3$ such that $\mathbf{C}^2 \subset \mathbf{C}^3$ is defined by $z = 0$. Then z^n is invariant under the action of G where n is the order of G. We claim that the principal divisor (z^n) on U is of the form

$$(z^n) = nY + \sum_j \frac{n(m_j - 1)}{m_j} B_j' + \sum_k d_k D_k \tag{4.3}$$

where $B_j', D_k \subset U$ are prime divisors such that $B_j' \cap Y = \tau_*^{-1} B_j$ and $D_k \cap Y$ is contained in the exceptional locus of $Y \to \mathbf{C}^2/G$ (or empty). This is saying that there exists an exceptional prime divisor B_j' of $U \to \mathbf{C}^3/G$ lying over B_j with $B_j' \cap Y = \tau_*^{-1} B_j$ and that its coefficient in (z^n) is $\frac{n(m_j-1)}{m_j}$. We may check this over the complete local ring $\hat{\mathcal{O}}_{\mathbf{C}^3/G, P}$ at a point $P \in B_j \backslash \{0\}$. Since G_j is the stabilizer subgroup of a point of $\mathbf{C}^3$ lying over P, there is an isomorphism of complete local rings:

$$\hat{\mathcal{O}}_{\mathbf{C}^3/G, P} \cong \hat{\mathcal{O}}_{\mathbf{C}^3/G_j, [0]}.$$

Let $\tilde{B}_j$ be a line in $\mathbf{C}^2$ mapped to B_j and take a G_j-invariant linear subspace $\tilde{B}_j^\perp$ of $\mathbf{C}^3$ such that

$$\mathbf{C}^3 = \tilde{B}_j \times \tilde{B}_j^\perp.$$

Then $G_j \cong \mathbf{Z}/m_j\mathbf{Z}$ is a subgroup of $\{1\} \times \mathrm{SL}(\tilde{B}_j^\perp)$ and therefore we have

$$\mathbf{C}^3/G_j \cong \tilde{B}_j \times (\tilde{B}_j^\perp/G_j),$$

where $\tilde{B}_j^{\perp}/G_j$ is a rational double point of type A_{m_j-1}. Thus we can see that on the crepant resolution

$$U \times_{(\mathbb{C}^3/G)} \operatorname{Spec} \hat{\mathcal{O}}_{\mathbb{C}^3/G,P} \to \operatorname{Spec} \hat{\mathcal{O}}_{\mathbb{C}^3/G,P} \cong \operatorname{Spec} \hat{\mathcal{O}}_{\mathbb{C}^3/G_j,[0]},$$

there is a prime divisor $\hat{B}_j'$ with desired properties such that the coefficient of $\hat{B}_j'$ in the divisor (z^{m_j}) is $m_j - 1$. Since m_j divides n, this proves (4.3).

From (4.2) and (4.3), we obtain

$$K_Y + \tau_*^{-1}B \equiv -\sum \frac{d_k}{n}(D_k \cap Y).$$

Here, note that z^n is a regular function and therefore the coefficients in (4.3) are all non-negative. Especially, we have $d_k \geq 0$ for all k. This proves the assertion since $K_X + B \in \operatorname{Pic}(X) \otimes \mathbb{Q} = 0$ in (3.1).

5. Abelian group case

Let $G \subset \mathrm{GL}(2, \mathbb{C})$ be a finite abelian subgroup of order n. As in the previous section, we embed $G \subset \mathrm{GL}(2, \mathbb{C})$ into $\mathrm{SL}(3, \mathbb{C})$.

Theorem 5. *Conjecture 4 is true if G is abelian.*

Proof. It is sufficient to prove the "if" part by Proposition 1. Let $Y \to X = \mathbb{C}^2/G$ be a resolution which is dominated by $Y_{\max}$. By Proposition 2 below, there is a projective crepant resolution $U \to \mathbb{C}^3/G$ such that $Y \subset U$. Then [CI04] ensures that there is a generic parameter θ such that $U \cong \mathcal{M}_\theta(\mathbb{C}^3)$. Then $\mathcal{M}_\theta(\mathbb{C}^2)$ is isomorphic to Y by Lemma 1.

Before stating the proposition, we need some notation. We diagonalize G and write

$$g = \operatorname{diag}(\zeta_n^{a_g}, \zeta_n^{b_g})$$

for $g \in G$ where ζ_n is a primitive n-th root of unity. Put

$$N_2 := \mathbb{Z}^2 + \sum_{g \in G} \mathbb{Z} \cdot \frac{1}{n}(a_g, b_g),$$

$$N_3 := \mathbb{Z}^3 + \sum_{g \in G} \mathbb{Z} \cdot \frac{1}{n}(a_g, b_g, -a_g - b_g)$$

which are the lattices of one-parameter subgroups for the toric varieties $\mathbb{C}^2/G$ and $\mathbb{C}^3/G$ respectively. The *junior simplex* $\Delta \subset (N_3)_{\mathbb{R}}$ is the triangle with vertices e_1, e_2, e_3 where $\{e_1, e_2, e_3\}$ is the basis of $\mathbb{Z}^3$ with $e_1, e_2 \in \mathbb{Z}^2$. A crepant resolution U corresponds to a basic triangulation of Δ. For a basic triangulation Σ of Δ, let U_Σ be the corresponding crepant resolution.

Consider the natural projection

$$p_{12} : N_3 \to N_2$$

and put $\Delta' := p_{12}(\Delta) \cong \Delta$. Let $e_i' \in (\mathbb{R}_{\geq 0})e_i \cap N_2$ be the primitive vector and write $e_i = m_i e_i'$ for $i = 1, 2$. If $B_i \subset \mathbb{C}^2/G$ denote the divisor corresponding to e_i', then

$$B := \frac{m_1 - 1}{m_1} B_1 + \frac{m_2 - 1}{m_2} B_2$$

is the boundary divisor for the quotient $\mathbb{C}^2/G$. A resolution Y of $\mathbb{C}^2/G$ is given by choosing primitive vectors $v_0, v_1, \ldots, v_s$ of $(\mathbb{Z}_{\geq 0})^2 \cap N_2$ such that $v_0 = e_1'$, $v_s = e_2'$ and $\{v_{i-1}, v_i\}$ is a basis of N_2 for $i = 1, \ldots, s$. If E_i denotes the exceptional divisor corresponding to v_i for $i = 1, \ldots, s-1$, then the discrepancy a_i of E_i for the pair (X, B) is $\alpha_i + \beta_i - 1$ where $v_i = (\alpha_i, \beta_i)$. Therefore, Y is dominated by the maximal resolution $Y_{\max}$ of (X, B) if and only if all of $v_1, \ldots, v_{s-1}$ are in Δ'.

Let $G_{(1,0)} \subset G$ be the stabilizer subgroup of $(1, 0) \in \mathbb{C}^2 = \mathbb{C}^2 \times \{0\} \subset \mathbb{C}^3$. Then $G_{(1,0)}$ acts on $\{1\} \times \mathbb{C}^2 \cong \mathbb{C}^2$ as a subgroup of $\mathrm{SL}(2)$ and the quotient $(\{1\} \times \mathbb{C}^2)/G_{(1,0)}$ is a closed subvariety of $\mathbb{C}^3/G$. Let

$$W \to (\{1\} \times \mathbb{C}^2)/G_{(1,0)}$$

be the minimal resolution. Notice that W is contained in any crepant resolution U of $\mathbb{C}^3/G$ since $(\{1\} \times \mathbb{C}^2)/G_{(1,0)} \subset \mathbb{C}^3/G$ is transversal to the one-dimensional stratum $(\mathbb{C}^\times \times \{(0,0)\})/G$. Now we prove the following proposition. The surjectivity of the ample cones will be used in the proof of the main theorem.

Proposition 2. *Let $Y \to \mathbb{C}^2/G$ be a resolution dominated by $Y_{\max}$. Then there is a projective crepant resolution $U = U_\Sigma \to \mathbb{C}^3/G$ containing Y such that the restriction map* $\mathrm{Amp}(U) \to \mathrm{Amp}(W)$ *of the ample cones is surjective.*

Proof. Since Y is dominated by $Y_{\max}$, it is defined by primitive vectors $v_0, v_1, \ldots, v_s \in \Delta' \cap N_2$. Let $w_i \in \Delta \cap N_3$ be the unique lift of v_i. For a basic triangulation Σ of Δ, $U = U_\Sigma$ contains Y if and only if the points connected to e_3 in Σ are exactly $w_0, \ldots, w_s$.

We prove the assertion by the induction on the order $|G|$ of G. If $|G| = 1$, then there is nothing to prove. We consider the number

$$\nu := \#(\{w_0, \ldots, w_{s-1}\} \backslash \{e_1\}) \geq 0.$$

If $v = 0$, then s must be 1 and $w_0 = e_1$ is a primitive vector. Especially, $\{e_1, v_1\}$ is a basis of N_2. In this case, Δ has a unique basic triangulation Σ and $U_\Sigma \cong W \times \mathbb{C}$. Hence the restriction map $\mathrm{Amp}(U_\Sigma) \to \mathrm{Amp}(W)$ is an isomorphism.

Suppose $v > 0$. Let $w \in \{w_0, \dots, w_{s-1}\} \setminus \{e_1\}$ be a point such that the coefficient of e_3 in w is the smallest. Then w determines a *star subdivision* of $\Delta : \Delta = \bigcup_{i=1}^{3} \Delta_i$ where Δ_1, Δ_2, Δ_3 are the triangles we_2e_3, we_1e_3, we_1e_2 respectively. Note that either Δ_2 or Δ_3 may be degenerate, in which case we simply ignore the degenerate one in the sequel. This subdivision of Δ, which is denoted by Σ_0, determines a projective crepant birational morphism $U_{\Sigma_0} \to \mathbb{C}^3/G$ where U_{Σ_0} is a toric variety with at most Gorenstein quotient singularities. The choice of w implies that $w_0, \dots, w_s$ are in $\Delta_1 \cup \Delta_2$. Hence by the induction hypothesis, there are basic triangulations Σ_1 and Σ_2 of Δ_1 and Δ_2 respectively, which satisfy the following conditions: in $\Sigma_1 \cup \Sigma_2$, the vertices connected to e_3 are exactly $w_0, \dots, w_s$, the map $\mathrm{Amp}(U_{\Sigma_1}) \to \mathrm{Amp}(W)$ is surjective and $\mathrm{Amp}(U_{\Sigma_2})$ is non-empty. We choose an arbitrary basic triangulation Σ_3 of Δ_3 with non-empty $\mathrm{Amp}(U_{\Sigma_3})$. Combining the triangulations Σ_1, Σ_2 and Σ_3 together, we obtain a basic triangulation of Δ such that $U_\Sigma \supset Y$. Since $\Delta = \bigcup_{i=1}^{3} \Delta_i$ is a star subdivision, we see that $U_\Sigma \to U_{\Sigma_0}$ is a projective morphism and the map $\mathrm{Amp}(U_\Sigma) \to \mathrm{Amp}(U_{\Sigma_1})$ is surjective. Therefore, the morphism $U_\Sigma \to \mathbb{C}^3/G$ is also projective and $\mathrm{Amp}(U_\Sigma) \to \mathrm{Amp}(W)$ is surjective.

6. *G*-constellations on a variety

In the case of G-constellations for non-abelian $G \subset \mathrm{GL}(2, \mathbb{C})$, we shall use the iterated construction of moduli spaces for a normal subgroup of G as in [IINdC13]. In order to do so, we have to consider G-constellations on a variety, rather than an affine space. Especially, the space of stability parameters will be larger than the affine case in general.

Suppose U is a quasi projective variety of finite type over $\mathbb{C}$ and G is a finite group acting on U. Let $\mathrm{coh}^G(U)$ be the abelian category of G-equivariant coherent sheaves on U and $\mathrm{coh}^G_{\mathrm{cpt}}(U)$ its subcategory consisting of sheaves whose supports are proper over $\mathbb{C}$. The corresponding Grothendieck groups are denoted by $K(\mathrm{coh}^G(U))$ and $K(\mathrm{coh}^G_{\mathrm{cpt}}(U))$ respectively. We also consider the perfect derived category $\mathrm{Perf}^G(U)$ of G-equivariant perfect complexes and its Grothendieck group $K(\mathrm{Perf}^G(U))$. For $\alpha \in K(\mathrm{Perf}^G(U))$ and $\beta \in K(\mathrm{coh}^G_{\mathrm{cpt}}(U))$, we write

$$\chi(\alpha, \beta) := \sum_i (-1)^i \dim \mathrm{Ext}^i_{\mathcal{O}_U}(\alpha, \beta)^G. \tag{6.1}$$

Let $\mathrm{coh}^G_{0\text{-dim}}(U)$ be the subcategory of $\mathrm{coh}^G_{\mathrm{cpt}}(U)$ consisting of sheaves with 0-dimensional support. We define the stability condition of objects in $\mathrm{coh}^G_{0\text{-dim}}(U)$.

Definition 4. Fix a class $\xi \in K(\mathrm{Perf}^G(U))$. An object $E \in \mathrm{coh}^G_{0\text{-dim}}(U)$ is said to be *ξ-stable* (or *ξ-semistable*) if $\chi(\xi, E) = 0$ and if for every non-trivial G-equivariant subsheaf F of E, $\chi(\xi, [F]) > 0$ (or $\chi(\xi, [F]) \geq 0$).

In the case where $U = \mathbb{C}^N$ is an affine space with a linear G-action, $K(\mathrm{Perf}^G(U)) = K(\mathrm{coh}^G(U))$ is isomorphic to (the dual of) the representation ring $R(G)$ and the definition coincides with the ($\mathbb{Z}$-valued) one in §2.1.

We have a well-defined function $\mathrm{rank} : K(\mathrm{Perf}^G(U)) \to \mathbb{Z}$ which extends the rank of a locally free sheaf. Put

$$K(\mathrm{Perf}^G(U))^0 := \{\xi \in K(\mathrm{Perf}^G(U)) \mid \mathrm{rank}\,\xi = 0\}.$$

Definition 5. A *G-constellation* on U is a G-equivariant coherent sheaf E on U with finite support such that $H^0(E)$ is isomorphic to the regular representation of G as a representation of G and $\chi(\xi, E) = 0$ for any $\xi \in K(\mathrm{Perf}^G(U))^0$.

For any $\xi \in K(\mathrm{Perf}^G(U))^0$, we can discuss the ξ-(semi)stabilities of G-constellations on U according to Definition 4. Since the multiplication by a positive integer does not change the stability condition, we may replace $K(\mathrm{Perf}^G(U))^0$ by $K(\mathrm{Perf}^G(U))^0_{\mathbb{Q}}$.

Remark 1. *In general, there may exist an object E supported on several fixed points such that $H^0(E) \cong R(G)$ but $\chi(\xi, E) \neq 0$ for some $\xi \in K(\mathrm{Perf}^G(U))^0$. Definition 5 excludes such cases.*

Remark 2. *If U is smooth, then $K(\mathrm{Perf}^G(U))$ coincides with $K(\mathrm{coh}^G(U))$ and we write $K(\mathrm{coh}^G(U))^0$ instead of $K(\mathrm{Perf}^G(U))^0$.*

Now we define the moduli functors of G-constellations:

Definition 6. Fix a class $\xi \in K(\mathrm{Perf}^G(U))^0_{\mathbb{Q}}$. Then the moduli functor for the ξ-stable G-constellations on U is defined to be the functor

$$S \mapsto \{\text{flat families of } \xi\text{-stable } G\text{-constellations parameterized by } S\}/\sim$$

for a locally noetherian scheme S over $\mathbb{C}$ where $E_S \sim F_S$ for flat families E_S and F_S means that there is a line bundle L on S such that $E_S \cong F_S \otimes L$.

Remark 3. *We show the existence of the moduli scheme in a very special case in Theorem 6. We do not discuss the existence problem in a general case in this paper.*

7. Iterated construction of moduli spaces

In this section, let V denote either $\mathbb{C}^2$ or $\mathbb{C}^3$ and consider a finite subgroup $G \subset \mathrm{GL}(V)$ with a normal subgroup N of G such that $N \subset \mathrm{SL}(V)$. Let

$$\theta^N : R(N) \to \mathbb{Z}$$

be a generic stability parameter for N-constellations on V, which is fixed by the conjugate action of G on $R(N)$. Put $Y_N = \mathcal{M}_{\theta^N}(V)$ and $\bar{G} = G/N$. Since $N \subset \mathrm{SL}(V)$ and $\dim V \leq 3$, there is an equivalence

$$\Phi : D^b(\mathrm{coh}^{\bar{G}}(Y_N)) \cong D^b(\mathrm{coh}^G(V)) \tag{7.1}$$

as in [IU15, Theorem 4.1] defined by

$$\Phi(-) = \mathbb{R}(p_V)_*((p_{Y_N})^*(-) \otimes \mathcal{U})$$

where p_V, p_{Y_N} are the projections of $Y_N \times V$ and $\mathcal{U}$ is the universal family of N-constellations.

Lemma 2. *Let $\mathcal{E}$ be a $\bar{G}$-equivariant coherent sheaf on Y_N with finite support. Then $\mathcal{E}$ is a $\bar{G}$-constellation on Y_N if and only if $\Phi(\mathcal{E})$ is a G-constellation on V. In this case, $\Phi(\mathcal{E})$ is θ^N-semistable.*

Proof. By the definition of Φ, we can see that $\Phi(\mathcal{E})$ is a 0-dimensional sheaf. Since Φ is an equivalence, we have $\chi(\xi, \mathcal{E}) = \chi(\Phi(\xi), \Phi(\mathcal{E}))$. Moreover, we can see $\operatorname{rank} \xi = \operatorname{rank} \Phi(\xi)$ for any $\xi \in K(\mathrm{coh}^{\bar{G}}(Y_N))$. Therefore, if $\mathcal{E}$ is a $\bar{G}$-constellation, $\chi(\xi, \Phi(\mathcal{E})) = 0$ for any $\xi \in K(\mathrm{coh}^G(V))^0$. This implies that $H^0(\Phi(\mathcal{E}))$ is a multiple of the regular representation $\mathbb{C}[G]$. If we regard $\mathcal{E}$ as an object of $\mathrm{coh}(Y_N)$, it is an Artinian sheaf of length $|\bar{G}|$ and therefore $\Phi(\mathcal{E})$ as an object of $\mathrm{coh}^N(V)$ has a filtration of length $|\bar{G}|$ whose factors are θ^N-stable N-constellations. Therefore, $\Phi(\mathcal{E})$ is θ^N-semistable and $H^0(\Phi(\mathcal{E}))$ as a representation of N is the direct sum of $|\bar{G}|$ copies of the regular representation of N. This implies that $H^0(\Phi(\mathcal{E})) \cong \mathbb{C}[G]$ and therefore $\Phi(\mathcal{E})$ is a G-constellation. The converse is proved in the same way.

The following lemma follows from the arguments in [BKR01, §8]:

Lemma 3. *Let E be an N-equivariant coherent sheaf on V with finite support such that $H^0(E)$ is isomorphic to $\mathbb{C}[N]^{\oplus s}$ for some integer $s > 0$ as a $\mathbb{C}[N]$-module. If E is θ^N-stable, then we have $s = 1$, i.e., E is an N-constellation.*

We compose θ^N with the restriction map $R(G) \to R(N)$ and regard it as a stability parameter for G-constellations as in [IINdC13, §2.2].

LEMMA 4. *Let E be a G-equivariant coherent sheaf on V with finite support such that $H^0(E) \cong \mathbb{Z}[G]^{\oplus s}$ for some s. If E is θ^N-semistable in* $\mathrm{coh}^G(V)$*, then it is also θ^N-semistable in* $\mathrm{coh}^N(V)$.

PROOF. Let $\eta : R(N) \to \mathbb{Z}$ be a group homomorphism such that $\eta(\rho) > 0$ for any irreducible representation ρ of N. We further suppose η is invariant under the conjugate action of G. Then,

$$Z(E) := \theta^N(H^0(E)) + \sqrt{-1}\eta(H^0(E))$$

defines a G-invariant Bridgeland stability condition [Bri07, Example 5.5] (see also [BCZ17, Lemma 7.1.3]) on $\mathrm{coh}^N(V)_0$, the category of N-equivariant coherent sheaves on V with 0-dimensional support. As in [BCZ17, Lemma 7.1.5], the equality $\theta^N(H^0(E)) = 0$ implies that E is θ^N-semistable if and only if it is semistable with respect to Z. Assume E is not θ^N-semistable and let $F \subset E$ be the first step of the Harder-Narasimhan filtration of E in $\mathrm{coh}^N(E)$ with respect to Z. Then the uniqueness of the HN filtration and the G-invariance of Z imply that F is invariant under the G-action. This means that F is a subsheaf of E in $\mathrm{coh}^G(V)$, which contradicts the θ^N-semistability of E in $\mathrm{coh}^G(V)$.

PROPOSITION 3. *The functor Φ induces a bijection from the set of $\bar{G}$-constellations on Y_N to the set of θ^N-semistable G-constellations on V.*

PROOF. If $\mathscr{E}$ is a $\bar{G}$-constellation on Y_N, then $\Phi(\mathscr{E})$ is a θ^N-semistable G-constellation by Lemma 2. Conversely, suppose E is a θ^N-semistable G-constellation on V. By Lemma 2, it suffices to show that $\Phi^{-1}(E)$ lies in $\mathrm{coh}^{\bar{G}}(Y_N)$ and has a 0-dimensional support. For this purpose, we may regard Φ as an equivalence $D^b(\mathrm{coh}\, Y_N) \cong D^b(\mathrm{coh}^N(V))$. By Lemma 4, E is θ^N-semistable as a sheaf in $\mathrm{coh}^N(V)$ and therefore has a filtration whose factors are θ^N-stable N-constellations by Lemma 3. Then, $\Phi^{-1}(E)$ as an object in $D^b(\mathrm{coh}(Y_N))$ is a sheaf with a filtration whose factors are skyscraper sheaves. This is what we needed.

Let

$$\varphi : K(\mathrm{coh}^{\bar{G}}(Y_N))^0_{\mathbb{Q}} \xrightarrow{\sim} K(\mathrm{coh}^G(V))^0_{\mathbb{Q}} \cong \Theta$$

be the isomorphism induced by Φ. The following theorem generalizes [IINdC13, Theorem 2.6].

THEOREM 6. *Let $\theta^N : R(N) \to \mathbb{Z}$ be a generic stability condition for N-constellations fixed by the conjugate action of G and $\xi \in K(\mathrm{coh}^{\bar{G}}(Y_N))^0$ be a stability parameter for $\bar{G}$-constellations on Y_N.*

(1) *There exists a scheme* $\mathscr{M}_\xi(Y_N)$ *representing the moduli functor for* ξ*-stable* $\bar{G}$*-constellations on* Y_N.

(2) *If we put*

$$\theta := m\theta^N + \varphi(\xi)$$

for $m \gg 0$, *then* $\mathscr{M}_\theta(V)$ *is isomorphic to the moduli space* $\mathscr{M}_\xi(Y_N)$ *of* ξ*-stable* $\bar{G}$*-constellations on* Y_N.

Proof. What we prove is that $\mathscr{M}_\theta(V)$ in (2) represents the moduli functor in (1). We choose m so that

$$m > \sum_{\rho \in \mathrm{Irr}(G)} |(\varphi(\xi))(\rho)| \dim \rho.$$

Then for any subsheaf F of a G-constellation, we have $|(\varphi(\xi))(F)| < m$.

Let $\mathscr{E}$ be a ξ-stable $\bar{G}$-constellation on Y_N. Then $\Phi(\mathscr{E})$ is a θ^N-semistable G-constellation by Proposition 3. Therefore, a subsheaf F of $\Phi(\mathscr{E})$ satisfies $\theta^N(F) \geq 0$. If $\theta^N(F) > 0$, then we have $\theta(F) > 0$ by our choice of m. If $\theta^N(F) = 0$, then there is a subsheaf $\mathscr{F}$ of $\mathscr{E}$ such that $F = \Phi(\mathscr{F})$ as in [IINdC13, Lemma 2.6]. Then we obtain $\theta(F) = \chi(\xi, \mathscr{F}) > 0$ by the ξ-stability of $\mathscr{E}$. Thus $\Phi(\mathscr{E})$ is θ-stable.

Conversely, suppose E is a θ-stable G-constellation on V. Then it is θ^N-semistable by our choice of m and therefore $\mathscr{E} := \Phi^{-1}(E)$ is a $\bar{G}$-constellation by Proposition 3. For a subsheaf $\mathscr{F} \subset \mathscr{E}$, $F := \Phi(\mathscr{F})$ has a filtration as an object of $\mathrm{coh}^N(V)$ whose factors are N-constellations. Therefore F satisfies $\theta^N(F) = 0$ and hence we obtain $\chi(\xi, \mathscr{F}) = \theta(F) > 0$, which proves the ξ-stability of $\mathscr{F}$.

Thus we have a bijection between ξ-stable $\bar{G}$-constellations and θ-stable G-constellations. To establish an isomorphism $\mathscr{M}_\theta(V) \cong \mathscr{M}_\xi(Y_N)$, we show that for any locally noetherian scheme S over $\mathbb{C}$, this bijection can be extended to a bijection between flat families of ξ-stable $\bar{G}$-constellations and flat families of θ-stable G-constellations parameterized by S. Let $\mathscr{U}$ be the universal N-constellation on $Y_N \times V$ and $\mathscr{U}_S$ be the pull back of $\mathscr{U}$ to $Y_N \times V \times S$. Then we can define a functor

$$\Phi_S : D^b(\mathrm{coh}^{\bar{G}}\ Y_N \times S) \to D^b(\mathrm{coh}^G\ V \times S)$$

by

$$\Phi_S(-) = \mathbb{R}(p_{V \times S})_*(\mathscr{U}_S \otimes p^*_{Y_N \times S}(-))$$

whose quasi-inverse is given by

$$\Phi_S^{-1}(-) = ((p_{Y_N \times S})_*(\mathscr{U}_S^\vee[\dim V] \overset{\mathbb{L}}{\otimes} p^*_{V \times S}(-))^N.$$

Suppose $\mathscr{E}_S$ is a flat family of ξ-stable $\bar{G}$-constellations on Y_N parameterized by S. Then, for any geometric point s of S, we have $\Phi_S(\mathscr{E}_S) \overset{\mathbb{L}}{\otimes} \mathcal{O}_s \cong \Phi(\mathscr{E}_s)$ as in [Bri99, Lemma 4.1], which is a θ-stable G-constellation on V. Hence the argument in [Bri99, Proposition 4.2] implies that $\Phi_S(\mathscr{E}_S)$ is actually a flat family of G-constellations on V. Conversely, if E_S is a flat family of θ-stable G-constellations, the same argument shows that $\Phi_S^{-1}(E_S)$ is a flat family of ξ-stable N-constellations on Y_N.

8. The case $G \ni -I$

In this section, put $V = \mathbb{C}^2$ and assume that $G \subset \mathrm{GL}(V)$ contains $-I$, where I is the identity matrix. We put $N := \langle -I \rangle \subset G$ and $\bar{G} := G/N$. Let θ^N be any generic stability parameter for N-constellations (which is automatically fixed by the conjugate action of G since N is central) and let $Y_N = \mathscr{M}_{\theta^N}(V)$ be the moduli space of N-constellations on V, on which $\bar{G}$ acts naturally. Since Y_N is a crepant resolution of the A_1 singularity V/N, the maximal resolution of $(Y_N/\bar{G}, B_N)$ coincides with the maximal resolution of (X, B), where B_N is the boundary divisor on Y_N determined by the ramification of $Y_N \to Y_N/\bar{G}$.

Let C be the exceptional curve of $Y_N \to V/N$. Then the equivalence (7.1) restricts to the equivalence

$$\Phi : D^b(\mathrm{coh}_C^{\bar{G}}(Y_N)) \cong D^b(\mathrm{coh}_0^G(V)) \tag{8.1}$$

of full subcategories consisting of objects supported by the subsets $C \subset Y_N$ and $\{0\} \subset V$ respectively. Consider the Grothendieck groups of (8.1):

$$K(\mathrm{coh}_C^{\bar{G}}(Y_N)) \cong K(\mathrm{coh}_0^G(V)), \tag{8.2}$$

where $K(\mathrm{coh}_0^G(V))$ is isomorphic to the representation ring $R(G)$ of G. Recall that there is a perfect pairing

$$\chi : K(\mathrm{coh}^G(V)) \times K(\mathrm{coh}_0^G(V)) \to \mathbb{Z}$$

defined by (6.1), which is isomorphic to

$$\chi : K(\mathrm{coh}^{\bar{G}}(Y_N)) \times K(\mathrm{coh}_C^{\bar{G}}(Y_N)) \to \mathbb{Z}$$

by Φ. Let

$$F_i K(\mathrm{coh}_C^{\bar{G}}(Y_N)) \subset K(\mathrm{coh}_C^{\bar{G}}(Y_N))$$

be the subgroup generated by the classes of objects whose supports are at most i-dimensional. Then the classes of $\bar{G}$-constellations on Y_N lie in

$F_0K(\mathrm{coh}_C^{\bar{G}}(Y_N))$ and for a stability parameter

$$\xi \in K(\mathrm{coh}^{\bar{G}}(Y_N))_{\mathbb{Q}} \cong K(\mathrm{coh}_C^{\bar{G}}(Y_N))_{\mathbb{Q}}^*,$$

the actual stability condition depends only on its image in $F_0K(\mathrm{coh}_C^{\bar{G}}(Y_N))_{\mathbb{Q}}^*$. In the next subsection, we investigate the structure of $F_0K(\mathrm{coh}_C^{\bar{G}}(Y_N))$.

8.1. Structure of $F_0K(\mathrm{coh}_C^{\bar{G}}(Y_N))$. In this subsection, we assume that G is not abelian. Notice that G acts on the exceptional curve $C \cong \mathbb{P}(V)$ through the homomorphism

$$G \hookrightarrow \mathrm{GL}(V) \twoheadrightarrow \mathrm{PGL}(V)$$

and let $Z \subset G$ be the kernel of $G \to \mathrm{PGL}(V)$. It is the subgroup consisting of scalar matrices in G.

Since G is non-abelian, $G/Z \subset \mathrm{PGL}(V)$ is a polyhedral (or dihedral) group acting on $\mathbb{P}(V)$ which we regard as a (real) 2-sphere. There are three non-free G/Z-orbits in C: the projections of the vertices, edges and faces of the regular polyhedron to the sphere. These orbits are denoted by O_1, O_2 and O_3 respectively.

For a $\bar{G}$-orbit $O \subset C$, let $\mathrm{coh}_O^{\bar{G}}(Y_N)$ denote the category of $\bar{G}$-equivariant coherent sheaves supported on O. Then we have an equivalence

$$\mathrm{coh}_O^{\bar{G}}(Y_N) \cong \mathrm{coh}_P^{\bar{G}_P}(Y_N) \tag{8.3}$$

where $\bar{G}_P$ is the stabilizer subgroup of a point $P \in O$ and $\mathrm{coh}_P^{\bar{G}_P}(Y_N)$ is the category of $\bar{G}_P$-equivariant coherent sheaves supported on P. Taking the Grothendieck groups of the both sides, we obtain

$$K(\mathrm{coh}_O^{\bar{G}}(Y_N)) \cong R(\bar{G}_P) \tag{8.4}$$

where $R(\bar{G}_P)$ is the representation ring of $\bar{G}_P$ regarded as an additive group.

Let $\bar{G}_k \subset \bar{G}$ be the stabilizer subgroup of a point in O_k, which is an abelian group since $\bar{Z} := Z/N \subset \bar{G}_k$ is central and $\bar{G}_k/\bar{Z}$ is cyclic. We consider the pushforward maps

$$K(\mathrm{coh}_{O_k}^{\bar{G}}(Y_N)) \to F_0K(\mathrm{coh}_C^{\bar{G}}(Y_N)) \tag{8.5}$$

for $k = 1, 2, 3$. By (8.4) for $O = O_k$, these maps are regarded as maps

$$\beta_k : R(\bar{G}_k) \to F_0K(\mathrm{coh}_C^{\bar{G}}(Y_N)).$$

Since $\bar{Z}$ is a subgroup of $\bar{G}_k$, we have the induction maps

$$\alpha_k : R(\bar{Z}) \to R(\bar{G}_k).$$

Define a map $\alpha : R(\bar{Z})^{\oplus 2} \to R(\bar{G}_1) \oplus R(\bar{G}_2) \oplus R(\bar{G}_3)$ by

$$\alpha(a,b) = (\alpha_1(a), -\alpha_2(a) + \alpha_2(b), -\alpha_3(b)).$$

The purpose of this subsection is to prove the following.

Proposition 4. *Let $\bar{G}_k$, β_k, α be as above. Then the following is an exact sequence of additive groups:*

$$0 \to R(\bar{Z})^{\oplus 2} \xrightarrow{\alpha} R(\bar{G}_1) \oplus R(\bar{G}_2) \oplus R(\bar{G}_3) \xrightarrow{\beta} F_0 K(\mathrm{coh}_C^{\bar{G}}(Y_N)) \to 0$$

where $\beta = (\beta_1, \beta_2, \beta_3)$.

The proof of the proposition is divided into three steps below. We first show that β is surjective:

Step 1. *The additive group $F_0 K(\mathrm{coh}_C^{\bar{G}}(Y_N))$ is generated by sheaves supported on $O_1 \cup O_2 \cup O_3$.*

Proof. It is obvious that $F_0 K(\mathrm{coh}_C^{\bar{G}}(Y_N))$ is generated by simple objects (objects having no non-trivial subobjects). Moreover, a simple object is supported on a single orbit O and is determined by an irreducible representation of the stabilizer subgroup $\bar{G}_P$ of a point $P \in O$ by (8.3). Therefore, it is sufficient to show that the class in $K(\mathrm{coh}_C^{\bar{G}}(Y_N))$ of a simple object $\mathscr{E}$ supported on a free G/Z-orbit O_f coincides with the class of some object $\mathscr{F}$ supported on $O_1 \cup O_2 \cup O_3$. Actually, we prove that for any $k \in \{1,2,3\}$ we can choose such an object $\mathscr{F}$ supported on O_k. Simple objects supported on the orbit O_f are determined by irreducible representations of the stabilizer subgroup $\bar{Z} \subset \bar{G}$ by (8.3). To describe them, notice that $C = \mathbb{P}(V)$ carries a G-equivariant line bundle $\mathscr{L} = \mathscr{O}_C(1)$ on which an element $\lambda I \in Z$ acts as the fiber-wise scalar multiplication by λ. On $\mathscr{L}^2$, the G-action is reduced to a $\bar{G}$-action and the induced actions of $\bar{Z}$ on the fibers of $\mathscr{L}^0, \mathscr{L}^2, \dots, \mathscr{L}^{2(l-1)}$ are the irreducible representations of the cyclic group $\bar{Z}$, where l is the order of $\bar{Z}$. Therefore, the simple objects supported on O_f are

$$\mathscr{L}^0|_{O_f}, \mathscr{L}^2|_{O_f}, \dots, \mathscr{L}^{2(l-1)}|_{O_f}, \tag{8.6}$$

where we regard O_f as a reduced subscheme. Now consider the exact sequences

$$0 \to \mathscr{L}^{2i} \otimes \mathscr{O}_C(-O_f) \to \mathscr{L}^{2i} \to \mathscr{L}^{2i}|_{O_f} \to 0$$

and

$$0 \to \mathscr{L}^{2i} \otimes \mathscr{O}_C(-n_k O_k) \to \mathscr{L}^{2i} \to \mathscr{L}^{2i}|_{n_k O_k} \to 0$$

for any $k \in \{1,2,3\}$ where n_k is the order of $\bar{G}_k/\bar{Z}$. If we show $\mathcal{O}_C(-O_f) \cong \mathcal{O}_C(-n_k O_k)$ in $\mathrm{coh}^{\bar{G}}(Y_N)$, then we obtain

$$[\mathcal{L}^{2i}|_{O_f}] = [\mathcal{L}^{2i}|_{n_k O_k}] \tag{8.7}$$

in $K(\mathrm{coh}_C^{\bar{G}}(Y_N))$ for any k as desired.

Finally, we show $\mathcal{O}_C(-O_f) \cong \mathcal{O}_C(-n_k O_k)$. Let $\bar{C} \cong \mathbb{P}^1$ be the quotient of C by the action of G/Z. Then both $\mathcal{O}_C(-O_f)$ and $\mathcal{O}_C(-n_k O_k)$ are the pull-backs of $\mathcal{O}_{\bar{C}}(-1)$ (equipped with the trivial $\bar{G}$-action) and hence we obtain the isomorphism.

Step 2. $\beta \circ \alpha = 0$.

Proof. This is equivalent to the equality

$$\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2 = \beta_3 \circ \alpha_3.$$

We recall the isomorphism (8.4) for a free G/Z-orbit $O_f \subset C$:

$$R(\bar{Z}) \cong K(\mathrm{coh}_{O_f}^{\bar{G}}(Y_N)).$$

Then it is sufficient to prove that $\beta_k \circ \alpha_k$ is identified with the pushforward map

$$K(\mathrm{coh}_{O_f}^{\bar{G}}(Y_N)) \to F_0 K(\mathrm{coh}_C^{\bar{G}}(Y_N)).$$

Recall that $K(\mathrm{coh}_{O_f}^{\bar{G}}(Y_N))$ has a basis of the form (8.6) and that their images in $K(\mathrm{coh}_C^{\bar{G}}(Y_N))$ satisfy (8.7). Hence the problem is reduced to showing that the map

$$K(\mathrm{coh}_{O_f}^{\bar{G}}(Y_N)) \to K(\mathrm{coh}_{O_k}^{\bar{G}}(Y_N))$$

defined by

$$[\mathcal{L}^{2i}|_{O_f}] \mapsto [\mathcal{L}^{2i}|_{n_k O_k}]$$

is identified with the induction map α_k. The irreducible representation ρ_i of $\bar{Z}$ corresponding to $[\mathcal{L}^{2i}|_{O_f}]$ is defined by sending $[\lambda I] \in \bar{Z}$ to $\lambda^{2i} \in \mathbb{C}^\times$. On the other hand, we have

$$[\mathcal{L}^{2i}|_{n_k O_k}] = \sum_{j=0}^{n_k-1} [\mathcal{L}^{2i}(-jO_k)|_{O_k}].$$

Here $\mathcal{L}^{2i}|_{O_k}$ corresponds to a representation of $\bar{G}_k$ whose restriction to $\bar{Z}$ is ρ_i. Moreover, $\mathcal{O}_C(-jO_k)|_{O_k}$ $(0 \le j \le n_k - 1)$ correspond to the irreducible representations of the cyclic group $\bar{G}_k/\bar{Z}$. Thus the element of $R(\bar{G}_k)$ corresponding to $[\mathcal{L}^{2i}|_{n_k O_k}]$ is the sum of all the irreducible representations of $\bar{G}_k$ whose restrictions to $\bar{Z}$ are ρ_i. Since $\bar{G}_k$ is an abelian group, this is the induced representation of ρ_i. Thus we obtain $\beta \circ \alpha = 0$.

Step 3. $\ker \beta = \mathrm{Im}\, \alpha$.

Proof. Notice that coker α is torsion free, β is surjective and $\beta \circ \alpha = 0$. Therefore it suffices to show

$$\operatorname{rank} F_0 K(\operatorname{coh}_C^{\bar{G}}(Y_N)) = \sum_{k=1}^{3} \operatorname{rank} R(\bar{G}_k) - 2 \operatorname{rank} R(\bar{Z}).$$

This follows from the following two equalities:

$$\operatorname{rank} F_0 K(\operatorname{coh}_C^{\bar{G}}(Y_N)) = \operatorname{rank} R(G) - \operatorname{rank} R(\bar{Z}) \tag{8.8}$$

$$\sum_{k=1}^{3} \operatorname{rank} R(\bar{G}_k) = \operatorname{rank} R(G) + \operatorname{rank} R(\bar{Z}). \tag{8.9}$$

We first consider (8.8). The isomorphism (8.2) reduces (8.8) to the equality

$$\operatorname{rank} K(\operatorname{coh}_C^{\bar{G}}(Y_N))/F_0 K(\operatorname{coh}_C^{\bar{G}}(Y_N)) = \operatorname{rank} R(\bar{Z})$$

and therefore it suffices to show that the classes

$$[\mathcal{O}_C], [\mathcal{L}^2], \dots, [\mathcal{L}^{2(l-1)}] \tag{8.10}$$

form a free basis of the quotient $K(\operatorname{coh}_C^{\bar{G}}(Y_N))/F_0 K(\operatorname{coh}_C^{\bar{G}}(Y_N))$ where

$$l := \operatorname{rank} R(\bar{Z}) = |\bar{Z}|.$$

Recall that $\mathcal{L}^2 \cong \omega_C^{-1}$ is a $\bar{G}$-equivariant line bundle on $C = \mathbb{P}(V)$. Since $\bar{Z}$ acts on C trivially, if we regard $\mathcal{L}^2$ as an object of $\operatorname{coh}^{\bar{Z}}(C)$, we have

$$\mathcal{L}^{2i} \cong \mathcal{O}_C(2i) \otimes \rho_{\bar{\imath}} \qquad \text{in } \operatorname{coh}^{\bar{Z}}(C) \tag{8.11}$$

where $\bar{\imath} = i \bmod l$ and $\rho_0, \rho_1, \dots, \rho_{l-1}$ are the irreducible representations of the cyclic group $\bar{Z} \cong \mathbb{Z}/l\mathbb{Z}$. This implies that (8.10) is linearly independent. To see that (8.10) is a generator, we show that for any object $\mathcal{E} \in \operatorname{coh}_C^{\bar{G}}(Y_N)$ its class $[\mathcal{E}]$ is a linear combination of (8.10) modulo $F_0 K(\operatorname{coh}_C^{\bar{G}}(Y_N))$. We may assume that $\mathcal{E}$ is a locally free sheaf on C and we use the induction on rank $\mathcal{E}$. If rank $\mathcal{E} = 0$, there is nothing to prove and we may suppose rank $\mathcal{E} > 0$. If we regard $\mathcal{E}$ as an object of $\operatorname{coh}^{\bar{Z}}(C)$, it splits as $\mathcal{E} = \bigoplus_i \mathcal{E}_i \otimes_{\mathbb{C}} \rho_i$ with $\mathcal{E}_i \in \operatorname{coh}(C)$. Suppose $\mathcal{E}_i \neq 0$. For any integer m we have

$$\operatorname{Hom}_{\mathcal{O}_C}(\mathcal{L}^{2i}, \mathcal{E} \otimes \mathcal{L}^{2lm})^{\bar{G}} = H^0((\mathcal{E} \otimes \mathcal{L}^{2ml-2i})^{\bar{Z}})^{\bar{G}/\bar{Z}}. \tag{8.12}$$

Here, (8.11) shows

$$(\mathcal{E} \otimes \mathcal{L}^{2ml-2i})^{\bar{Z}} \cong \mathcal{E}_i \otimes \mathcal{O}(2ml - 2i) \neq 0$$

and the restriction map

$$H^0((\mathcal{E} \otimes \mathcal{L}^{2ml-2i})^{\bar{Z}}) \to H^0((\mathcal{E} \otimes \mathcal{L}^{2ml-2i})^{\bar{Z}}|_{O_f})$$

is surjective for a $\bar{G}/\bar{Z}$-free orbit $O_f \subset C$ if m is sufficiently large. Since $H^0((\mathscr{E} \otimes \mathscr{L}^{2ml+2i})^{\bar{Z}}|_{O_f})$ is a non-zero multiple of the regular representation of $\bar{G}/\bar{Z}$, its $\bar{G}/\bar{Z}$-invariant part is non-zero. Therefore, (8.12) is non-zero and hence there is a non-zero homomorphism

$$\alpha : \mathscr{L}^{2i} \hookrightarrow \mathscr{E} \otimes \mathscr{L}^{2lm}.$$

Now the induction hypothesis shows that coker α is a linear combination of (8.10) modulo $F_0K(\mathrm{coh}_C^{\bar{G}}(Y_N))$. This shows that the class $[\mathscr{E} \otimes \mathscr{L}^{2lm}]$ is also a linear combination of (8.10) modulo $F_0K(\mathrm{coh}_C^{\bar{G}}(Y_N))$. Since we have

$$[\mathscr{E}] - [\mathscr{E} \otimes \mathscr{L}^{2lm}] \in F_0K(\mathrm{coh}_C^{\bar{G}}(Y_N)),$$

$[\mathscr{E}]$ is a linear combination of (8.10) modulo $F_0K(\mathrm{coh}_C^{\bar{G}}(Y_N))$. Thus (8.10) is a free basis of $K(\mathrm{coh}_C^{\bar{G}}(Y_N))/F_0K(\mathrm{coh}_C^{\bar{G}}(Y_N))$ and therefore we have established (8.8).

Next we prove (8.9). Let $\mathrm{ZL}(V) \subset \mathrm{GL}(V)$ be the subgroup consisting of the non-zero scalar matrices and consider the multiplication map

$$\mu : \mathrm{ZL}(V) \times \mathrm{SL}(V) \to \mathrm{GL}(V).$$

Then the kernel of μ is a group of order 2 generated by $(-I, -I)$. We put $\tilde{G} = \mu^{-1}(G)$ and let $H \subset \mathrm{SL}(V)$ be the image of $\tilde{G}$ with respect to the second projection. For any element $(z, h) \in \tilde{G}$, denote by $Z_{\tilde{G}}(z, h)$ and $Z_G(zh)$ the centralizers of (z, h) in $\tilde{G}$ and zh in G respectively. Then the restriction $\mu : Z_{\tilde{G}}(z, h) \to Z_G(zh)$ is a surjective two-to-one map and hence the number of conjugates of (z, h) coincides with the number of conjugates of zh. Therefore, the number of conjugacy classes in $\tilde{G}$ is twice the number of conjugacy classes in G. Thus we obtain

$$\mathrm{rank}\ R(G) = \frac{1}{2}\,\mathrm{rank}\ R(\tilde{G}).$$

Moreover, since $\tilde{G}/Z \cong H$ and Z is central in $\tilde{G}$, this can be written as

$$\mathrm{rank}\ R(G) = \frac{1}{2}\,\mathrm{rank}\ R(H) \times |Z| = \mathrm{rank}\ R(H) \times |\bar{Z}|. \tag{8.13}$$

Notice that H acts on V and $\bar{H} := H/N \cong \bar{G}/\bar{Z} \subset \mathrm{PGL}(V)$ acts on $C = \mathbb{P}(V)$. Since H is in $\mathrm{SL}(V)$, the McKay correspondence for the binary polyhedral (or dihedral) group H establishes

$$\sum_{k=1}^{3} |\bar{H}_k| = \mathrm{rank}\ R(H) + 1 \tag{8.14}$$

where $\bar{H}_k \subset \bar{H}$ is the stabilizer of a point in O_k (the left hand side of (8.14) is two plus the number of the irreducible exceptional curves in the minimal resolution of V/H, which is also the minimal resolution of $Y_N/\bar{H}$). Moreover, the isomorphism $\bar{H} \cong \bar{G}/\bar{Z}$ implies

$$|\bar{H}_k| \times |\bar{Z}| = |\bar{G}_k| = \text{rank } R(\bar{G}_k). \tag{8.15}$$

Putting the equalities (8.13), (8.14) and (8.15) together, we obtain (8.9).

Corollary 1. *The dual module* $\mathrm{Hom}_{\mathbf{Z}}(F_0K(\mathrm{coh}_C^{\bar{G}}(Y_N)), \mathbf{Z})$ *is isomorphic to*

$$\left\{(\theta_1, \theta_2, \theta_3) \in \bigoplus_{k=1}^{3} \mathrm{Hom}_{\mathbf{Z}}(R(\bar{G}_k), \mathbf{Z}) \,\middle|\, \theta_1|_{\bar{Z}} = \theta_2|_{\bar{Z}} = \theta_3|_{\bar{Z}}\right\}.$$

8.2. Main theorem.

Proposition 5. *Suppose a finite subgroup* $G \subset \mathrm{GL}(2, \mathbf{C})$ *contains* $-I$ *and* $Y \to Y_N/\bar{G}$ *is a resolution dominated by* $Y_{\max}$. *Then there exists a generic stability parameter* $\theta \in \Theta$ *such that* $\mathscr{M}_\theta \cong Y$. *Especially, the maximal resolution* $Y_{\max}$ *of* $(\mathbf{C}^2/G, B)$ *is isomorphic to the moduli space of G-constellations for some generic stability parameter* θ.

Proof. We may assume G is non-abelian by Theorem 5 so we may apply the results of section 8.1. If we show there exists a generic parameter $\xi \in K(\mathrm{coh}^{\bar{G}}(Y_N))^0_{\mathbf{Q}}$ such that $\mathscr{M}_\xi(Y_N) \cong Y$, then the assertion follows from Theorem 6.

Let $P \in C$ be a point. Since $\bar{G}$ acts on $Y_N \times \mathbf{C} = \mathscr{M}_{\theta^N}(V \times \mathbf{C})$ and $\bar{Z}$ fixes $(P, 0)$, $\bar{Z}$ acts on the Zariski tangent space $\tilde{T} := T_{(P,0)}(Y_N \times \mathbf{C}) \cong \mathbf{C}^3$ as a subgroup of $\mathrm{SL}(\tilde{T})$. Note that as a representation of $\bar{Z}$, $\tilde{T}$ is independent of the choice of the point P. Let $T' \subset \tilde{T}$ be the two-dimensional $\bar{Z}$-invariant subspace transversal to C; then $\bar{Z} \subset \mathrm{SL}(T')$. Fix a generic stability parameter $\theta^{\bar{Z}} \in R(\bar{Z})^*_{\mathbf{Q}}$ for $\bar{Z}$-constellations (on $\tilde{T}$) satisfying $\theta^{\bar{Z}}(\mathbf{C}[\bar{Z}]) = 0$. Then $W := \mathscr{M}_{\theta^{\bar{Z}}}(T')$ is the minimal resolution of $T'/\bar{Z}$. The Fourier-Mukai transform

$$\varphi^*_{\theta^{\bar{Z}}} : R(\bar{Z})^*_{\mathbf{Q}} \cong K(\mathrm{coh}^{\bar{Z}}(T'))_{\mathbf{Q}} \xrightarrow{\sim} K(\mathrm{coh}\, W)_{\mathbf{Q}}$$

sends $\theta^{\bar{Z}}$ to an element $l_{\theta^{\bar{Z}}}$ of $F^1K(\mathrm{coh}\, W)_{\mathbf{Q}} \cong \mathrm{Pic}(W)_{\mathbf{Q}}$ and it lies in the ample cone $\mathrm{Amp}(W)$ as in (2.1). (Notice that here $\dim T' = 2$ and $F^2K(\mathrm{coh}\, W) = 0$.)

Take a point P_k in the orbit O_k for each $k \in \{1, 2, 3\}$. We consider the tangent spaces $\tilde{T}_k := T_{(P_k,0)}(Y_N \times \mathbf{C})$ and $T_k = T_{P_k}(Y_N)$. Let R_k denote the complete local ring of $T_k/\bar{G}_k$ at $[0]$ which is isomorphic to the complete local ring of $Y_N/\overline{G}$ at $[P_k]$:

$$R_k := \hat{\mathcal{O}}_{T_k/\bar{G}_k,[0]} \cong \hat{\mathcal{O}}_{Y_N/\bar{G},[P_k]}.$$

By this isomorphism, there is a resolution

$$Y_k \to T_k/\overline{G}_k$$

with an isomorphism

$$Y_k \times_{(T_k/\bar{G}_k)} \operatorname{Spec} R_k \cong Y \times_{(Y_N/\bar{G})} \operatorname{Spec} R_k \tag{8.16}$$

over $\operatorname{Spec} R_k$. Since $\overline{G}_k$ is abelian, we can apply Proposition 2 where the first factor of $T_k \cong \mathbf{C}^2$ is $T_{P_k}(C)$ (so that $(1,0)$ lies in $T_{P_k}(C)$ and $G_{(1,0)} = \overline{Z}$) and obtain a projective crepant resolution

$$U_{\Sigma_k} \to \tilde{T}_k/\overline{G}_k$$

such that $Y_k \subset U_{\Sigma_k}$ and that the restriction map $\operatorname{Amp}(U_{\Sigma_k}) \to \operatorname{Amp}(W)$ is surjective. Choose a class $l_k \in \operatorname{Amp}(U_{\Sigma_k})$ which is mapped to $l_{\theta^{\bar{Z}}} \in \operatorname{Amp}(W)$ for each k. Then by Theorem 2 we can find a generic stability parameter θ_k for $\overline{G}_k$-constellations on $\tilde{T}_k$ such that $\mathcal{M}_{\theta_k}(\tilde{T}_k) \cong U_{\Sigma_k}$ and the class of $\varphi_{\theta_k}^*(\theta_k)$ in $\operatorname{Pic}(U_{\Sigma_k})_{\mathbf{Q}}$ coincides with l_k. Since $[\varphi_{\theta_k}^*(\theta_k)] = l_k$ and l_k restricts to $l_{\theta^{\bar{Z}}}$, θ_k restricts to $\theta^{\overline{Z}}$ on $R(\overline{Z})$. Then Corollary 1 shows that $(\theta_1, \theta_2, \theta_3)$ determines an element of $F_0K(\operatorname{coh}_C^{\overline{G}}(Y_N))_{\mathbf{Q}}^*$. Lift it to an element $\xi \in K(\operatorname{coh}^{\overline{G}}(Y_N))_{\mathbf{Q}} \cong K(\operatorname{coh}_C^{\overline{G}}(Y_N))_{\mathbf{Q}}^*$. Since the restriction of ξ to $K(\operatorname{coh}^{\overline{G}}(O_k))_{\mathbf{Q}} \cong R(\overline{G}_k)_{\mathbf{Q}}^*$ is θ_k which is of rank 0, we have rank $\xi = 0$ and we can consider the moduli space $\mathcal{M}_\xi(Y_N)$.

We claim that there is an isomorphism

$$\mathcal{M}_\xi(Y_N) \times_{(Y_N/\bar{G})} \operatorname{Spec} R_k \cong \mathcal{M}_{\theta_k}(T_k) \times_{(T_k/\bar{G}_k)} \operatorname{Spec} R_k \tag{8.17}$$

over $\operatorname{Spec} R_k$. For any locally noetherian scheme S over $\operatorname{Spec} R_k$, an S-valued point of the left hand side of (8.17) is given by a flat family of ξ-stable $\overline{G}$-constellations on Y_k parameterized by S, which is an object of $\operatorname{coh}^{\overline{G}}(Y_N \times_{(Y_N/\bar{G})} S)$. Similarly, an S-valued point of the right hand side of (8.17) is given by a flat family of θ_k-stable $\overline{G}_k$-constellations on T_k parameterized by S, which is an object of $\operatorname{coh}^{\overline{G}_k}(T_k \times_{(T_k/\bar{G}_k)} S)$.

Notice that

$$\begin{aligned} Y_N \times_{(Y_N/\bar{G})} S &\cong (Y_N \times_{(Y_N/\bar{G})} \operatorname{Spec} R_k) \times_{(\operatorname{Spec} R_k)} S \\ &\cong \left(\coprod_{Q \in O_k} \operatorname{Spec} \hat{\mathcal{O}}_{Y_N,Q} \right) \times_{(\operatorname{Spec} R_k)} S \\ &\supset \operatorname{Spec} \hat{\mathcal{O}}_{Y_N,P_k} \times_{(\operatorname{Spec} R_k)} S \end{aligned}$$

$$\cong \operatorname{Spec} \hat{\mathcal{O}}_{T_k,0} \times_{(\operatorname{Spec} R_k)} S$$
$$\cong T_k \times_{(T_k/\bar{G}_k)} S$$

which induces an equivalence

$$\operatorname{coh}^{\bar{G}}(Y_N \times_{(Y_N/\bar{G})} S) \cong \operatorname{coh}^{\bar{G}_k}(T_k \times_{(T_k/\bar{G}_k)} S)$$

(this is almost the same as (8.3)). This equivalence gives a bijection between S-valued points of the both sides of (8.17) and we obtain (8.17).

Our choice of θ_k implies $\mathcal{M}_{\theta_k}(T_k) \cong Y_k$ and hence (8.16) and (8.17) yield an isomorphism

$$\mathcal{M}_\xi(Y_N) \times_{(Y_N/\bar{G})} \operatorname{Spec} R_k \cong Y \times_{(Y_N/\bar{G})} \operatorname{Spec} R_k.$$

over Spec R_k. Since $\mathcal{M}_\xi(Y_N)$ and Y are both isomorphic to $Y_N/\bar{G}$ except over the points $[P_1]$, $[P_2]$, and $[P_3]$, we obtain $\mathcal{M}_\xi(Y_N) \cong Y$.

Recall that we say $G \subset \mathrm{GL}(2,\mathbb{C})$ is *small* if G acts freely on $\mathbb{C}^2 \backslash \{0\}$. The following lemma follows from the classification of small subgroups of $\mathrm{GL}(2,\mathbb{C})$ but we give a proof for the reader's sake.

Lemma 5. *If a finite small subgroup $G \subset \mathrm{GL}(2,\mathbb{C})$ is non-abelian, then it contains $-I$ as a unique element of order* 2.

Proof. If G is non-abelian, then its image $G' \subset \mathrm{PGL}(2,\mathbb{C})$ is also non-abelian and therefore it is either a dihedral or a polyhedral group. Especially, the orders $|G'|$ and $|G|$ are even. Then G contains an element of order 2. If it is not $-I$, then it fixes a line in $\mathbb{C}^2$, contradicting the smallness of G.

Theorem 7. *If $G \subset \mathrm{GL}(2,\mathbb{C})$ is a finite small subgroup, then Conjecture* 4 *is true.*

Proof. The abelian case follows from Theorem 5. Otherwise, G contains $-I$ by the above lemma. Moreover, the minimal resolution of V/G factors through $Y_N/\bar{G}$; see [Bri68]. Then the assertion follows from Proposition 5.

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