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Twisted cohomology pairings of knots III; triple cup products

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ABSTRACT. Given a representation of a link group, we introduce a trilinear form as a topological invariant. We show that, if the link is either hyperbolic or a knot with the malnormal peripheral subgroup, then the trilinear form is equal to the pairing of the (twisted) triple cup product and the fundamental relative 3-class. We give some examples illustrating the main results.

1. Introduction

This paper examines knot invariants of trilinear forms, while the previous papers [N2, N3] in this series discussed those of bilinear forms. In general, the bilinear form arising from the Poincaré duality is a powerful tool, as in algebraic surgery theory. In contrast, there are relatively fewer studies of trilinear forms. However, some 3-forms and trilinear cup products appear in 3-dimensional geometry together with topological information (see, e.g., [CGO, M, L, S, Tur]).

We give the definition of a trilinear pairing (see (1)) in a general situation where the coefficients are arbitrary. Let Y be a compact orientable 3-manifold with toroidal boundary and with a fixed fundamental class $[Y, \partial Y] \in H_3(Y, \partial Y; \mathbb{Z}) \cong \mathbb{Z}$. Take a group homomorphism $f : \pi_1(Y) \to G$, a right G-module M, and a G-invariant trilinear function $\psi : M^3 \to A$ over a ring A. Then, we can define the composite map

$$H^1(Y, \partial Y; M)^{\otimes 3} \xrightarrow{\smile} H^3(Y, \partial Y; M^{\otimes 3}) \xrightarrow{\langle \bullet, [Y, \partial Y] \rangle} M^{\otimes 3} \xrightarrow{\psi} A. \qquad (1)$$

Here M is regarded as the local coefficient of Y via f, and the first map $\smile$ is the cup product, and the second is defined by the pairing with $[Y, \partial Y]$. However, the trilinear form (1) is considered to be something uncomputable. Actually, it seems hard to concretely express the 3-class $[Y, \partial Y]$ and the cup product.

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This paper addresses the case where Y is the 3-manifold which is obtained from the 3-sphere by removing an open tubular neighborhood of a link L, i.e., $Y = S^3 \backslash \nu L$. We show that, if L is a hyperbolic link, we obtain a diagrammatic method of computing the trilinear pairings. To be precise, starting from a diagram of L, we define an invariant of trilinear form, and show that the invariant is equal to the trilinear form (1), if L is a hyperbolic link (Theorem 2.5). In addition, we also show a similar theorem in the torus knot case (see Theorem 2.6). The point in the theorems is that, in the computation of (1), we do not need to describe $[Y, \partial Y]$ and cup products, thus, this computation is not so hard; see the examples in Section 4. As an application, when Y is a 3-fold covering space of S^3 branched along a hyperbolic link L and M is a trivial coefficient, we give a diagrammatic computation of the trilinear form (1) (Theorem 3.1).

This paper is organized as follows. Section 2 formulates the trilinear forms in terms of the quandle cocycle invariants, and states the main theorems. Section 3 discusses a relationship to 3-fold branched coverings. Section 4 describes some computations. Section 5 gives the proofs of the theorems.

Notation. The symbol L denotes a smoothly embedded oriented link in the 3-sphere S^3. We write E_L for the 3-manifold which is obtained from S^3 by removing an open tubular neighborhood of L.

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2. Results: diagrammatic formulations of the trilinear forms

Our purpose in this section is to give a trilinear form invariant (Theorem 2.3), and to state the main results in §2.2. For this purpose, §2.1 starts by reviewing colorings, and formulates some link-invariants of trilinear forms.

Thorough this section, we fix a group G and a right G-module M over a ring A.

2.1. Preliminary: formulation of the first cohomology. We need some notation from [IIJO, N2] before proceeding. Denote $M \times G$ by X, and define a binary operation on X by

$$\lhd : (M \times G) \times (M \times G) \to M \times G, \qquad (a, g, b, h) \mapsto ((a - b) \cdot h + b, h^{-1}gh), \quad (2)$$

which was first introduced in [IIJO, Lemma 2.2], and satisfies "the quandle axiom". We fix a link $L \subset S^3$ with a group homomorphism $f : \pi_1(S^3 \backslash L) \to G$.

Next, we review colorings. Choose an oriented diagram D of L. Then, it follows from the Wirtinger presentation of D that the homomorphism f is regarded as a map $\{\text{arcs of } D\} \to G$. A map $\mathscr{C} : \{\text{arcs of } D\} \to X$ is an *X-coloring* if it satisfies $\mathscr{C}(\alpha_\tau) \lhd \mathscr{C}(\beta_\tau) = \mathscr{C}(\gamma_\tau)$ at each crossings of D as illustrated in Figure 1. It is worth noticing that the set of all colorings is regarded as a subset of the direct product X^{α_D}, where α_D is the number of the arcs of D. Let $\mathrm{Col}_X(D_f)$ denote the set of all X-colorings over f, that is,

$$\mathrm{Col}_X(D_f) := \{\mathscr{C} \in (M \times G)^{\alpha_D} \mid \mathscr{C} \text{ is an } X\text{-coloring, } p_G \circ \mathscr{C} = f\}, \tag{3}$$

where p_G is the projection $X = M \times G \to G$. Then, we can easily verify from the linear operation (2) that $\mathrm{Col}_X(D_f)$ is made into an abelian subgroup of $M^{\alpha(D)}$, and that the diagonal subset $M_{\mathrm{diag}} \subset M^{\alpha_D}$ is a direct summand in $\mathrm{Col}_X(D_f)$. Denoting the other summand by $\mathrm{Col}_X^{\mathrm{red}}(D_f)$, we have a decomposition

$$\mathrm{Col}_X(D_f) \cong \mathrm{Col}_X^{\mathrm{red}}(D_f) \oplus M_{\mathrm{diag}}.$$

The previous paper [N2] gave a topological meaning of the coloring sets as follows:

Theorem 2.1 ([N2]). *Let E_L be a link complement in S^3. Regard the G-module M as a local system of E_L via $f : \pi_1(E_L) \to G$. Then, there are isomorphisms*

$$\mathrm{Col}_X(D_f) \cong H^1(E_L, \partial E_L; M) \oplus M, \qquad \mathrm{Col}_X^{\mathrm{red}}(D_f) \cong H^1(E_L, \partial E_L; M). \tag{4}$$

Let us review shadow colorings [CKS, IIJO]. A *shadow coloring* is a pair of a coloring $\mathscr{C}$ over f and a map λ from the complementary regions of D to M, satisfying the condition depicted in the right side of Figure 1 for every arc. Let $\mathrm{SCol}_X(D_f)$ denote the set of shadow colorings of D such that the unbounded exterior region is assigned by $0 \in M$. Notice that, by the coloring rules, assignments of the other regions are uniquely determined from the unbounded region, and admit, therefore, a shadow coloring; we thus obtain

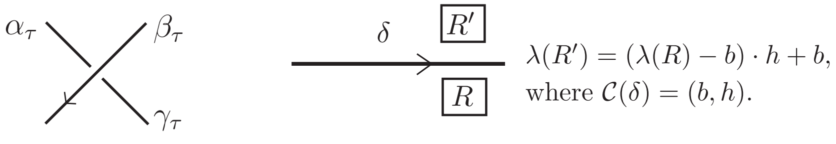

Fig. 1. The coloring conditions at each crossing τ and around each arcs.

a bijection

$$\mathrm{Col}_X(D_f) \simeq \mathrm{SCol}_X(D_f). \tag{5}$$

2.2. Invariants of trilinear forms. We will explain Definition 2.2 below, and show Theorem 2.3.

For this, we need two things: first, we take three G-modules M_1, M_2, M_3 and the associated $X_i = M_i \times G$. Let A be an abelian group. Let us consider a trilinear map $\psi : M_1 \times M_2 \times M_3 \to A$ over $\mathbb{Z}$ satisfying the G-invariance, that is,

$$\psi(a_1 \cdot g, a_2 \cdot g, a_3 \cdot g) = \psi(a_1, a_2, a_3), \tag{6}$$

holds for any $a_i \in M_i$ and $g \in G$.

Next, let us consider the map $X_1 \times X_2 \times X_3 \to A$ by the formula

$$((b_1, g_1), (b_2, g_2), (b_3, g_3)) \mapsto \psi((b_1 - b_2) \cdot (1 - g_2), b_2 - b_3, b_3 - b_3 \cdot g_3^{-1}), \tag{7}$$

for $a_i \in M_i$ and $g_1, g_2, g_3 \in G$. This map was first defined in [N1, Corollary 4.6]. Given three shadow colorings $\mathscr{S}_i \in \mathrm{SCol}_{X_i}(D_f)$ with $i \leq 3$ and each crossing τ of D, we can find assignments as illustrated in Figure 2. Inspired by the formula (7), we define a weight of τ to be

$$\mathscr{W}_{\psi,\tau}(\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3) := \psi((a_1 - b_1)(1 - g^{\varepsilon_\tau}), b_2 - c_2, c_3 - c_3 \cdot h^{-1}) \in A,$$

where $\varepsilon_\tau \in \{\pm 1\}$ is the sign of τ.

Definition 2.2. Given a G-invariant trilinear map $\psi : M_1 \times M_2 \times M_3 \to A$, we define a trilinear map

$$\mathscr{T}_\psi : \prod_{i=1}^{3} \mathrm{SCol}_{X_i}(D_f) \to A; \qquad (\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3) \mapsto \sum_\tau \mathscr{W}_{\psi,\tau}(\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3),$$

where τ runs over all the crossings of D.

The point is that, given a diagram D, we can diagrammatically deal with the trilinear map $\mathscr{T}_\psi$ by definitions; see §4.1–§4.3 for examples.

Next, we now show the invariance of $\mathscr{T}_\psi$ up to trilinear equivalence:

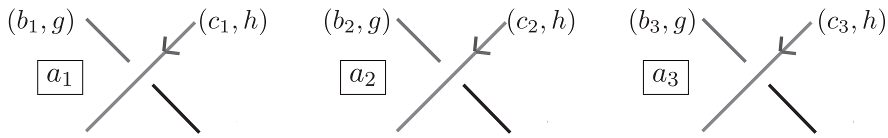

Fig. 2. Colors around a crossing with respect to the shadow colorings $\mathscr{S}_1$, $\mathscr{S}_2$, $\mathscr{S}_3$. Here (b_i, g) and (c_i, h) lie in $M \times G$.

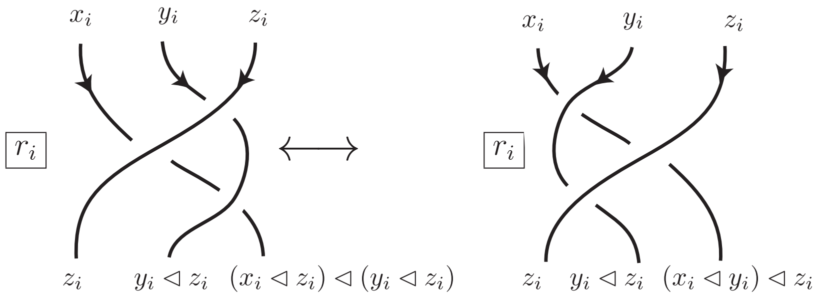

Fig. 3. The 1 : 1-correspondence associated with a Reidemeister move of type III.

Theorem 2.3. *Let two diagrams D and D' differ by a Reidemeister move. Then, there is a canonical isomorphism $\mathscr{B}_i : \mathrm{SCol}_{X_i}(D_f) \simeq \mathrm{SCol}_{X_i}(D'_f)$, for which the equality $\mathscr{T}_\psi = \mathscr{T}'_\psi \circ (\mathscr{B}_1 \otimes \mathscr{B}_2 \otimes \mathscr{B}_3)$ holds. In particular, the equivalence class of the trilinear map $\mathscr{T}_\psi$ depends only on the homomorphism $f : \pi_1(S^3 \backslash L) \to G$ and the input data (M_1, M_2, M_3, ψ).*

Proof. We first focus on the Reidemeister move of type III; see Figure 3. Considering the correspondence in Figure 3 with $x_i, y_i, z_i \in X_i$, we can construct the bijection $\mathscr{B}_i$, where $i \in \{1, 2, 3\}$. We suppose that the left region is colored by $r_i \in M$. Let us show the desired equality $\mathscr{T}_\psi = \mathscr{T}'_\psi \circ (\mathscr{B}_1 \otimes \mathscr{B}_2 \otimes \mathscr{B}_3)$. For this, take $a_i, b_i, c_i \in M_i$ and $g, h, k \in G$ such that $x_i = (a_i, g)$, $y_i = (b_i, h)$, $z_i = (c_i, k) \in X_i$. Then, the sum from the left side is, by definition and examining the figure, computed as

$$\begin{aligned}
&\psi((r_1 - a_1)(1 - g), a_2 - c_2, c_3(1 - k^{-1})) \\
&\quad + \psi((r_1 g - a_1 g + a_1 - b_1)(1 - h), b_2 - c_2, c_3(1 - k^{-1})) \\
&\quad + \psi((r_1 - a_1)k(1 - k^{-1}gk), (a_2 - b_2)k, (b_3 k - c_3 k + c_3)(1 - k^{-1}h^{-1}k)).
\end{aligned}$$

On the other hand, the sum from the right side is formulated as

$$\begin{aligned}
&\psi((r_1 - a_1)(1 - g), a_2 - b_2, b_3(1 - h^{-1})) \\
&\quad + \psi((r_1 - b_1)(1 - h), b_2 - c_2, c_3(1 - k^{-1})) \\
&\quad + \psi((r_1 - a_1)h(1 - h^{-1}gh), (b_2 - a_2)h + a_2 - c_2, c_3(1 - k^{-1})).
\end{aligned}$$

Then, an elementary calculation using (6) can show that the two sums are equal. However, since the calculation is a little tedious, we omit the details.

Finally, the required equality concerning Reidemeister moves of type I immediately follows from $\psi(0, y, z) = 0$, and the invariance under the Reidemeister move of type II is clear by a similar discussion.

REMARK 2.4. In this way, the construction for trilinear forms is applicable to not only tame links in S^3, but also handlebody-knots in S^3. In fact, by a similar discussion to [IIJO], we can easily check that the trilinear form is invariant with respect to the diagrammatic moves of handlebody-knots; see [IIJO, Figures 1 and 2] for the moves.

2.3. Topological meaning of the trilinear forms. As mentioned in the introduction, we will state (Theorems 2.5 and 2.6) that the trilinear forms of some links are equal to the trilinear pairings (The proofs of the theorems appear in §5).

THEOREM 2.5. *Let M_1, M_2, M_3 be G-modules as in Definition 2.2. Choose a fundamental class $[E_L, \partial E_L]$ in $H_3(E_L, \partial E_L; \mathbb{Z}) \cong \mathbb{Z}$. We assume that L is either a hyperbolic link or a prime knot which is neither a cable knot nor a torus knot. Then, via the identification* (4), *the trilinear form $\mathcal{T}_\psi$ is equal to the following composite map:*

$$\bigotimes_{i=1}^{3} H^1(E_L, \partial E_L; M_i) \xrightarrow{\smile} H^3(E_L, \partial E_L; M_1 \otimes M_2 \otimes M_3) \xrightarrow{\psi \circ \langle \bullet, [E_L, \partial E_L] \rangle} A. \quad (8)$$

The following theorem shows that a weak version of the identity in Theorem 2.5 also holds for the torus knots.

THEOREM 2.6. *Let M_1, M_2, M_3, ψ, and $[E_L, \partial E_L]$ be as above. Assume that L is the (m,n)-torus knot. Then, the trilinear form $\mathcal{T}_\psi$ is equal to the composite* (8) *modulo the integer $mn \in \mathbb{Z}$.*

As a concluding remark, while the triple cup product of a link is often considered to be speculative and uncomputable, it turns out to be computable only from a link diagram without describing $[E_L, \partial E_L]$ and any triangulation of $S^3 \backslash L$.

REMARK 2.7. We compare the trilinear forms in Definition 2.2 with the existing results on "the quandle cocycle invariants". Briefly speaking, the link invariant in [CKS] is constructed from a quandle X and a map $\Phi : X^3 \to A$ which satisfy "the quandle cocycle condition", and is defined to be a certain map $\mathcal{I}_\Phi : \mathrm{SCol}_X(D) \to A$. Then, our trilinear form is a trilinearization of the quandle cocycle invariants with respect to quandles of the form $X = M \times G$. To be precise, if $M = M_1 = M_2 = M_3$, we can see that the associated invariant $\mathcal{I}_\Phi : \mathrm{SCol}_X(D) \to A$ is equal to the composite $\mathcal{T}_\psi \circ (\triangle \times \mathrm{id}) \circ \triangle$ by definitions, where $\triangle$ is the diagonal map $\mathrm{SCol}_X(D) \to \mathrm{SCol}_X(D)^2$. In conclusion, the theorems also suggest topological meaning of the quandle cocycle invariants with $X = M \times G$.

REMARK 2.8. We also explain that the assumption for L in Theorem 2.6 stems from malnormality. Here, a subgroup $K \subset G$ is said to be *malnormal* if $g^{-1}Kg \cap K = \{1_G\}$ for any $g \in G \backslash K$. For a knot $K \subset S^3$, it is proved [HW, Wei] that the peripheral subgroup $\pi_1(\partial(S^3\backslash K)) \subset \pi_1(S^3\backslash K)$ is malnormal if and only if K is a prime knot which is neither a cable knot nor a torus knot, as above. Furthermore, for a hyperbolic link L, the peripheral subgroups of the link group is malnormal. Theorem 2.6 is proved in Section 5.2 by using the results in [N5], which in turn is proved by using the malnormality of peripheral subgroups.

3. Relation to 3-fold branched coverings

In this section, we consider a closed oriented 3-manifold N and the triple cup product

$$H^1(N; \mathbb{Z}/n\mathbb{Z})^{\otimes 3} \xrightarrow{\smile} H^3(N; \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\langle \bullet, [N] \rangle} \mathbb{Z}/n\mathbb{Z}, \tag{9}$$

where the coefficient module $\mathbb{Z}/n\mathbb{Z}$ is a G-module with the trivial G-action. Although there are studies of this map (see, e.g., [S, CGO, Tur]), there are few examples of computation. As an application of the theorems above, this section gives a recovery of the triple cup products of N, when N is a 3-fold cyclic covering of S^3 branched over a link.

To state Theorem 3.1, we need some terminology. Let G be $\mathbb{Z}/3\mathbb{Z} = \langle t \mid t^3 = 1 \rangle$. Consider the epimorphism $f : \pi_1(S^3\backslash L) \to G$ which sends every meridian to t, and the associated 3-fold cyclic branched covering $\tilde{C}_L \to S^3$.

THEOREM 3.1. *Let G be $\mathbb{Z}/n$, and the modules M_1, M_2, and M_3 in Definition 2.2 be $\mathbb{Z}[t^{\pm 1}]/(n, t^2 + t + 1)$ with $n \neq 3$. Let $p : \mathbb{Z}[t^{\pm 1}]/(n, t^2 + t + 1) \to \mathbb{Z}/n\mathbb{Z}$ be the epimorphism which sends $a + tb$ to a. Set up the map $\psi_0 : M^3 \to \mathbb{Z}/n\mathbb{Z}$ which takes (x, y, z) to xyz. As in Theorem 2.5, assume that L is either a hyperbolic link or a prime knot which is neither a cable knot nor a torus knot. Then, there is an isomorphism $\mathrm{Col}^{\mathrm{red}}_{X_i}(D_f) \cong H^1(\tilde{C}_L; \mathbb{Z}/n\mathbb{Z})$ such that the trilinear map $\mathscr{T}_{p \circ \psi_0}$ is equivalent to (9) with $N = \tilde{C}_L$.*

PROOF. We first show the isomorphism $\mathrm{Col}^{\mathrm{red}}_{X_i}(D_f) \cong H^1(\tilde{C}_L; \mathbb{Z}/n\mathbb{Z})$. Let R be the ring $\mathbb{Z}[t]/(n, t^2 + t + 1)$. By Theorem 2.1, we have $\mathrm{Col}^{\mathrm{red}}_{X_i}(D_f) \cong H^1(E_L; \partial E_L; M)$. Notice that $H^i(\partial E_L; M)$ is annihilated by $1 - t$. Since $1 - t$ and $1 + t + t^2$ are coprime, we have

$$H^1(E_L; \partial E_L; M) \cong H^1(E_L; M) \cong \mathrm{Hom}_{R\text{-mod}}(H_1(E_L, M), R). \tag{10}$$

Let $\tilde{E}_L \to E_L = S^3\backslash L$ be the 3-fold covering. Then, by Shapiro's lemma (see, e.g. [Bro]), the canonical inclusion $\imath : \mathbb{Z}/n \to R$ yields the isomorphisms:

$$H^*(\tilde{E}_L : \mathbb{Z}/n) \cong H^*(E_L; \mathbb{Z}[t]/(n, t^3 - 1))$$
$$\cong H^*(E_L; R) \oplus H^*(E_L; \mathbb{Z}[t]/(n, t-1)). \tag{11}$$

Here, the second isomorphism is obtained from the ring isomorphism $\mathbb{Z}[t]/(n, t^3 - 1) \cong R \oplus \mathbb{Z}[t]/(n, t-1)$. Let $i : \tilde{E}_L \hookrightarrow \tilde{C}_L$ be the inclusion. According to [Kaw, Theorem 5.5.1], the homology $H_1(\tilde{C}_L; \mathbb{Z})$ is annihilated by $1 + t + t^2$, and the induced map $i_* : H_1(\tilde{E}_L; \mathbb{Z}) \to H_1(\tilde{C}_L; \mathbb{Z})$ is a splitting surjection. Thus, the induced map $i^* : H^1(\tilde{C}_L; \mathbb{Z}/n) \to H^1(\tilde{E}_L; \mathbb{Z}/n)$ is injective and the image is isomorphic to $H^1(E_L; R)$. In summary, we obtained the desired isomorphism.

We will complete the proof. By (11), we have a splitting injection $\mathscr{S} : H^*(\tilde{E}_L, \partial\tilde{E}_L; R) \to H^*(E_L, \partial E_L; \mathbb{Z}/n)$. Take the canonical maps

$$j : (\tilde{E}_L, \partial\tilde{E}_L) \to (\tilde{C}_L, \tilde{C}_L \backslash \tilde{E}_L), \qquad \text{and} \qquad k : (\tilde{C}_L, \varnothing) \to (\tilde{C}_L, \tilde{C}_L \backslash \tilde{E}_L).$$

Then, we have the commutative diagram:

$$\begin{tikzcd}
H^1(E_L, \partial E_L; M)^{\otimes 3} \arrow[r, "\psi_0 \circ \smile"] \arrow[d, "\mathscr{S}"] & H^3(E_L, \partial E_L; M) \arrow[r, "\langle \bullet, [E_L, \partial E_L] \rangle"] \arrow[d, "\mathscr{S}"] & R \\
H^1(\tilde{E}_L, \partial\tilde{E}_L; \mathbb{Z}/n)^{\otimes 3} \arrow[r, "\smile"] & H^3(\tilde{E}_L, \partial\tilde{E}_L; \mathbb{Z}/n) \arrow[r, "\langle \bullet, [\tilde{E}_L, \partial\tilde{E}_L] \rangle"] & \mathbb{Z}/n \arrow[u, "\iota"'] \arrow[d, equal] \\
H^1(\tilde{C}_L, \tilde{C}_L \backslash \tilde{E}_L; \mathbb{Z}/n)^{\otimes 3} \arrow[r, "\smile"] \arrow[u, "\cong"', "j^*"] \arrow[d, "k^*"] & H^3(\tilde{C}_L, \tilde{C}_L \backslash \tilde{E}_L; \mathbb{Z}/n) \arrow[r, "\langle \bullet, [\tilde{C}_L, \tilde{C}_L \backslash \tilde{E}_L] \rangle"] \arrow[u, "\cong"', "j^*"] \arrow[d, "k^*"] & \mathbb{Z}/n \arrow[d, equal] \\
H^1(\tilde{C}_L; \mathbb{Z}/n)^{\otimes 3} \arrow[r, "\smile"] & H^3(\tilde{C}_L; \mathbb{Z}/n) \arrow[r, "\langle \bullet, [\tilde{C}_L] \rangle"] & \mathbb{Z}/n.
\end{tikzcd}$$

Here, the vertical maps j^* are the isomorphisms by the excision axiom. Moreover, by the discussion in the above paragraph, the composite $k^* \circ (j^*)^{-1} \circ \mathscr{S}$ is an isomorphism from $H^1(E_L, \partial E_L; M)$. Since $p \circ \iota : \mathbb{Z}/n \to \mathbb{Z}/n$ is an isomorphism, the following two composites are equivalent:

$$p \circ \psi_0 \circ \langle \bullet, [E_L, \partial E_L] \rangle \circ \smile, \qquad \langle \bullet, [\tilde{C}_L] \rangle \circ \smile.$$

By Theorem 2.5, the left hand side is equal to the trilinear map $\mathscr{T}_{p \circ \psi_0}$. Hence, $\mathscr{T}_{p \circ \psi_0}$ is equivalent to (9) with $N = \tilde{C}_L$ as desired.

4. Examples as diagrammatic computations

4.1. For the trefoil knot and the figure eight knot. We will compute the trilinear forms $\mathscr{T}_\psi$ associated with some homomorphisms $f : \pi_1(S^3 \backslash L) \to G$, where L is either the trefoil knot or the figure eight knot.

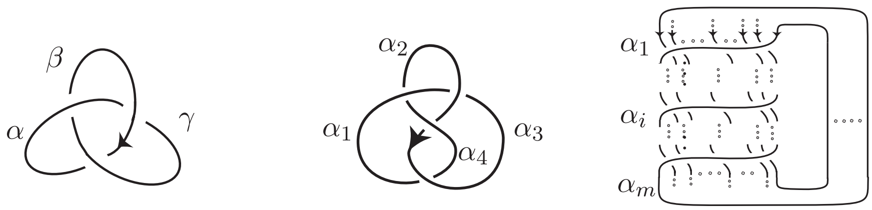

Fig. 4. The trefoil knot, the figure eight knot, and the (m, m)-torus knot.

As a simple example, we first focus on the trefoil knot 3_1. Let D be the diagram of K as illustrated in Figure 4. Note that the Wirtinger presentation is given by $\pi_1(S^3 \backslash L) \cong \langle \alpha, \beta \mid \alpha\beta\alpha = \beta\alpha\beta \rangle$. Then, we can easily see that a correspondence $\mathscr{C} : \{\alpha, \beta, \gamma\} \to X$ with

$$\mathscr{C}(\alpha) = (a_i, g), \qquad \mathscr{C}(\beta) = (b_i, g), \qquad \mathscr{C}(\alpha) = (c_i, g) \in M_i \times G$$

is an X-coloring over $f : \pi_1(S^3 \backslash L) \to G$, if and only if it satisfies the four equations

$$g = f(\alpha), \qquad h = f(\beta), \qquad ghg = hgh, \tag{12}$$

$$c_i = a_i \cdot h + b_i \cdot (1 - h), \tag{13}$$

$$(a_i - b_i) \cdot (1 - g + hg) = (a_i - b_i) \cdot (1 - h + gh) = 0. \tag{14}$$

Furthermore, given a G-invariant linear form ψ, the sum $\mathscr{T}_\psi$ is equal to

$$\begin{aligned} &\psi(-a_1 \cdot (1 - g), a_2 - b_2, a_3 \cdot (1 - h^{-1})) \\ &\quad + \psi(-b_1 \cdot (1 - h), b_2 - c_2, c_3 \cdot (1 - h^{-1}g^{-1}h)) \\ &\quad + \psi(-c_1 \cdot (1 - h^{-1}gh), c_2 - a_2, a_3 \cdot (1 - g^{-1})), \end{aligned}$$

by definition. Then, by canceling out c_i by using (12) and (13), we can easily obtain the following formula: for $((a_i, g), (b_i, h)) \in \mathrm{SCol}_{X_i}(D_f) \subset M_i^2$,

$$\mathscr{T}_\psi((a_1, b_1), (a_2, b_2), (a_3, b_3)) = \psi((a_1 - b_1)g^{-1}, (a_2 - b_2) \cdot h, a_3 - b_3) \in A. \tag{15}$$

Next, we will compute $\mathscr{T}_\psi$ of the figure eight knot. The computation can be done in a similar way to the trefoil case. We only describe the outline.

Let D be the diagram with four arcs as illustrated in Figure 4. Similarly, we can see that a correspondence $\mathscr{C} : \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \to X$ with $\mathscr{C}(\alpha_i) = (x_i, z_i) \in M_i \times G$ is an X-coloring $\mathscr{C}$ over $f : \pi_1(S^3 \backslash L) \to G$, if and only if it satisfies the following equations:

$$z_i = f(\alpha_i), \qquad z_2^{-1} z_1 z_2 = z_1^{-1} z_2^{-1} z_1 z_2 z_1^{-1} z_2 z_1 \in G, \tag{16}$$

$$x_3 = (x_1 - x_2) \cdot z_2 + x_2, \qquad x_4 = (x_2 - x_1) \cdot z_1 + x_1, \tag{17}$$

$$(x_1 - x_2) \cdot (z_1 + z_2 - 1) = (x_1 - x_2) \cdot (1 - z_2^{-1}) z_1 z_2$$
$$= (x_1 - x_2) \cdot (1 - z_1^{-1}) z_2 z_1 \in M. \tag{18}$$

Accordingly, it follows from (17) that the set $\mathrm{Col}_X(D_f)$ is generated by x_1, x_2.

Given a G-invariant trilinear form ψ, the trilinear from $\mathcal{T}_\psi$ is given by the following formula:

$$\mathcal{T}_\psi((x_1, x_2), (x_1', x_2'), (x_1'', x_2''))$$
$$= \psi((x_1 - x_2) \cdot z_1 z_2^{-1}, x_2' - x_1', (x_1'' - x_2'') \cdot (1 - z_2^{-1}))$$
$$+ \psi((x_1 - x_2) \cdot z_2^{-1} z_1, (x_1' - x_2') \cdot (1 - z_1), (x_1'' - x_2'') \cdot (1 - z_2^{-1}) z_1).$$

Remark 4.1. Unfortunately, the author does not know an example of (G, M_i, ψ) of a non-discrete Lie group G, G-modules M_i, and a G-invariant trilinear form, for which the trilinear form $\mathcal{T}_\psi$ on $M_1 \times M_2 \times M_3$ is nontrivial. Though he investigated the case where $G = SL_2(\mathbb{C})$ and $M = \mathbb{C}^2$ or $\mathbb{C}^3$, all resulting $\mathcal{T}_\psi$ turned out to be trivial. So, the author would like to propose it as an open problem to construct an example of (G, M_i, ψ), with G a non-discrete Lie group, which yields an nontrivial trilinear form $\mathcal{T}_\psi$.

4.2. The (m, m)-torus link $T_{m,m}$. We also calculate the trilinear form $\mathcal{T}_\psi$ concerning the (m, m)-torus link, following from Definition 2.2. These calculations is useful in the paper [N4], which suggests invariants of "Hurewitz equivalence classes".

Let L be the (m, m)-torus link $T_{m,m}$ with $m \geq 2$, and let $\alpha_1, \dots, \alpha_m$ be the arcs depicted in Figure 4. Let us identity α_{i+m} with α_i of period m. The Wirtinger presentation of $\pi_1(S^3 \backslash L)$ is

$$\langle a_1, \dots, a_m \,|\, a_1 \cdots a_m = a_m a_1 a_2 \cdots a_{m-1} = a_2 \cdots a_m a_1 \rangle.$$

Given a homomorphism $f : \pi_1(S^3 \backslash L) \to G$ with $f(\alpha_i) \in G$, let us discuss X-colorings $\mathcal{C}$ over f. Concerning the coloring condition on the ℓ-th link component, it satisfies the equation

$$(\cdots (\mathcal{C}(\alpha_\ell) \lhd \mathcal{C}(\alpha_{\ell+1})) \lhd \cdots) \lhd \mathcal{C}(\alpha_{\ell+m-1}) = \mathcal{C}(\alpha_\ell), \tag{19}$$

for any $1 \leq \ell \leq m$. With the notation $\mathcal{C}(\alpha_i) := (x_i, z_i) \in X$, this equation (19) reduces to a system of linear equations

$$(x_{\ell-1} - x_\ell) + \sum_{\ell \leq j \leq \ell+m-2} (x_j - x_{j+1}) \cdot z_{j+1} z_{j+2} \cdots z_{m+\ell} = 0 \in M, \tag{20}$$

for any $1 \leq \ell \leq m$. Conversely, we can easily verify that, if a map $\mathcal{C} : \{\text{arcs of } D\} \to X$ satisfies the equation (20), then $\mathcal{C}$ is an X-coloring.

Denoting the left side in (20) by $\Gamma_{f,\ell}(\vec{x})$, consider a homomorphism

$$\Gamma_f : M^m \to M^m; \qquad (x_1, \dots, x_m) \mapsto (\Gamma_{f,1}(\vec{x}), \dots, \Gamma_{f,m}(\vec{x})).$$

To conclude, the set $\mathrm{Col}_X(D_f)$ coincides with the kernel of Γ_f.

Next, we precisely describe the resulting trilinear form.

Proposition 4.2. *Let $f : \pi_1(S^3 \backslash T_{m,m}) \to G$ be as above. Let $\psi : M^3 \to A$ be a G-invariant linear functions. Then, the trilinear form $\mathscr{T}_\psi : \mathrm{Ker}(\Gamma_f)^{\otimes 3} \to A$ sends $(w_1, \dots, w_m) \otimes (x_1, \dots, x_m) \otimes (y_1, \dots, y_m)$ to*

$$\sum_{\ell=1}^{m} \sum_{k=1}^{m-1} \psi\left(w_\ell (1 - z_\ell) \hat{z}_{\ell+1;\ell+k-1}, \sum_{j=1}^{k} (x_{j+\ell-1} - x_{j+\ell}) \hat{z}_{j+\ell;k+\ell-1}, \right.$$
$$\left. y_{k+\ell}(1 - z_{k+\ell}^{-1}) \right). \tag{21}$$

Here, for $s \leq t$, we use the notations $\hat{z}_{s;t} := z_s z_{s+1} \cdots z_t$ and $\hat{z}_{s+1;s} := 1 \in G$.

The formula can be obtained by direct calculation.

4.3. Examples of Theorem 3.1. We will give some examples of Theorem 3.1. Under the setting of Theorem 3.1, let $G = \mathbb{Z}/3 = \langle t \mid t^3 = 1 \rangle$, and $f : \pi_1(S^3 \backslash L) \to \mathbb{Z}/3$ be the map which sends every meridian to t. Take $M_i = A = \mathbb{Z}[t]/(n, t^2 + t + 1)$ for some $n \in \mathbb{Z}_{\geq 0}$, and let $\psi_0 : M_1 \times M_2 \times M_3 \to A$ send (x, y, z) to xyz.

In this paragraph, we focus only on knots, K, such that $H^1(E_K, \partial E_K; A) \cong H^1(\tilde{B}_K; \mathbb{Z}/n)$ is isomorphic to either A or 0. We will write the trilinear map $\mathscr{T}_{\psi_0}$ as a cubic polynomial with respect to $(a, b, c) \in (H^1(E_K, \partial E_K; \mathbb{Z}/n))^3$. Then, we give the resulting computation of $\mathscr{T}_{\psi_0}$, when K is a prime knot with crossing number < 7. The list of the computation is given as follows:

Knot	n	$\mathscr{T}_{\psi_0}$
3_1	2	abc
4_1	4	$2abc$
5_1	any	0
5_2	5	$(1+t)abc$
6_1	any	0
6_2	any	0
6_3	any	0

5. Proofs of the theorems

We will complete the proofs of Theorems 2.5–2.6 in §5.3. While the statements were described in terms of ordinary cohomology, the proof will be done via the group cohomology. For this purpose, we review the relative group homology in §5.1.

5.1. Preliminary: Review of relative group cohomology. We will recall the relative group (co)homology in the non-homogeneous terms. Throughout this subsection, we fix a group Γ and a homomorphism $f : \Gamma \to G$. Then, we have the action of Γ on the right G-module M via f.

Set $C^n_{\mathrm{gr}}(\Gamma; M) = \mathrm{Map}(\Gamma^n, M)$. For $\phi \in C^n_{\mathrm{gr}}(\Gamma; M)$, define the coboundary $\partial^n(\phi) \in C^{n+1}_{\mathrm{gr}}(\Gamma; M)$ by the formula

$$\begin{aligned}\partial^n(\phi)(g_1, \dots, g_{n+1}) = \phi(g_2, \dots, g_{n+1}) \\ + \sum_{1 \le i \le n} (-1)^i \phi(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_{n+1}) \\ + (-1)^n \phi(g_1, \dots, g_n) g_{n+1}.\end{aligned}$$

Let K_j be a subgroup and $\iota_j : K_j \hookrightarrow \Gamma$ be the inclusion, where the index j runs over $1 \le j \le m$. Then, we can define the mapping cone of ι_j as follows:

$$C^n(\Gamma, K_{\mathscr{J}}; M) := \mathrm{Map}(\Gamma^n, M) \oplus \left(\bigoplus_j \mathrm{Map}((K_j)^{n-1}, M) \right).$$

For $(h, k_1, \dots, k_m) \in C^n(\Gamma, K_{\mathscr{J}}; M)$, define $\partial^n(h, k_1, \dots, k_m)$ in $C^{n+1}(\Gamma, K_{\mathscr{J}}; M)$ by

$$\begin{aligned}&\partial^n(h, k_1, \dots, k_m)(a, b_1, \dots, b_m) \\ &\quad = (\partial^n h(a), h(b_1) - \partial^{n-1} k_1(b_1), \dots, h(b_m) - \partial^{n-1} k_m(b_m)),\end{aligned}$$

where $(a, b_1, \dots, b_m) \in \Gamma^{n+1} \times K_1^n \times \cdots \times K_m^n$. Then, $(C^*(\Gamma, K_{\mathscr{J}}; M), \partial^*)$ is a cochain complex, and we can consider its cohomology.

We now observe the submodule consisting of 1-cocycles $Z^1(\Gamma, K_{\mathscr{J}}; M)$. Let us define the semi-direct product $M \rtimes G$ by

$$(a, g) \star (a', g') := (a \cdot g' + a', gg'), \qquad \text{for } a, a' \in M,\ g, g' \in G.$$

Let $\mathrm{Hom}_f(\Gamma, M \rtimes G)$ be the set of group homomorphisms $\Gamma \to M \rtimes G$ over the homomorphism f. Consider the following map:

$$\begin{aligned}Z^1(\Gamma, K_{\mathscr{J}}; M) \to \mathrm{Hom}_f(\Gamma, M \rtimes G) \oplus M^m; \\ (h, y_1, \dots, y_m) \mapsto (\gamma \mapsto (h(\gamma), f(\gamma)), y_1, \dots, y_m).\end{aligned}$$

Lemma 5.1 ([N2, Lemma 5.2]). *This map gives an isomorphism between* $Z^1(\Gamma, K_{\mathscr{J}}; M)$ *and the following subset of* $\mathrm{Hom}_f(\Gamma, M \rtimes G) \oplus M^m$*:*

$$\{(\tilde{f}, y_1, \dots, y_m) \,|\, \tilde{f}(h_j) = (y_j - y_j \cdot h_j, f_j(h_j)), \text{ for any } h_j \in K_j\}.$$

The image of ∂^1*, i.e.,* $B^1(\Gamma, K_{\mathscr{J}}; M)$*, is equal to the subset* $\{(\tilde{f}_a, a, \dots, a)\}_{a \in M}$*. Here, for* $a \in M$*, the map* $\tilde{f}_a : \Gamma \to M \rtimes G$ *is defined as the homomorphism which sends* γ *to* $(a - a \cdot \gamma, f(\gamma))$*. In particular, if* $K_{\mathscr{J}}$ *is not empty,* $B^1(\Gamma, K_{\mathscr{J}}; M)$ *is a direct summand of* $Z^1(\Gamma, K_{\mathscr{J}}; M)$*.*

Now, we review the cup product. When $K_{\mathscr{J}}$ is the empty set, the product of $u \in C^p(\Gamma; M)$ and $v \in C^q(\Gamma; M')$ is defined to be $u \smile v \in C^{p+q}(\Gamma; M \otimes M')$ given by

$$\begin{aligned}(u \smile v)(g_1, \dots, g_{p+q}) := (-1)^{pq}(u(g_1, \dots, g_p)g_{p+1} \cdots g_{p+q}) \\ \otimes v(g_{p+1}, \dots, g_{p+q}).\end{aligned} \tag{22}$$

When $K_{\mathscr{J}}$ is not empty, for two elements $(f, k_1, \dots, k_m) \in C^p(\Gamma, K_{\mathscr{J}}; M)$ and $(f', k_1', \dots, k_m') \in C^q(\Gamma, K_{\mathscr{J}}; M')$, let us define the *cup product* by

$$(f \smile f', k_1 \smile f', \dots, k_m \smile f') \in C^{p+q}(\Gamma, K_{\mathscr{J}}; M \otimes M').$$

This formula yields a bilinear map, by passage to cohomology.

Finally, we observe another complex. Let us define the module $C^n_{\mathrm{red}}(\Gamma)$ by the formula

$$\{(c_1, \dots, c_m) \in \mathrm{Map}(\mathbb{Z}[\Gamma^n], M)^m \,|\, c_1 + c_2 + \cdots + c_m = 0 \in \mathrm{Map}(\mathbb{Z}[\Gamma^n], M)\}.$$

Then, this complex canonically has an inclusion into the direct sum of $C^n(\Gamma, K_j)$:

$$P_n : C^n_{\mathrm{red}}(\Gamma) \to \bigoplus_{j: 1 \le j \le m} C^n(\Gamma, K_j).$$

Then, we define a quotient complex, $D^n(\Gamma, K_{\mathscr{J}}; M)$, to be the cokernel of P_n. Then, $C^n(\Gamma, K_{\mathscr{J}}; M)$ is isomorphic to $D^n(\Gamma, K_{\mathscr{J}}; M)$, because the kernel of the inclusions $\bigoplus_{j=1}^m C^n(\Gamma, K_j) \to C^n(\Gamma, \mathscr{K})$ is the image of P_n.

Remark 5.2. We give a natural relationship to the usual cohomology. Take the Eilenberg-MacLane spaces of type $(\Gamma, 1)$ and of type $(K_j, 1)$, and consider the map $(\iota_j)_* : K(K_j, 1) \to K(\Gamma, 1)$ induced by the inclusions. Then the relative homology $H_n(\Gamma, K_{\mathscr{J}}; M)$ is isomorphic to the homology of the mapping cone of $\bigsqcup_j K(K_j, 1) \to K(\Gamma, 1)$ with local coefficients. Further, the cup product $\smile$ above coincides with that on the singular cohomology groups.

We mention the case where L is either a knot or an unsplittable link. (We note that any hyperbolic knot is unsplittable). Then, the complement $S^3\backslash L$ is an Eilenberg-MacLane space by the sphere theorem. Since we only use Γ as $\pi_1(S^3\backslash L)$ in this paper, we may discuss only the relative group cohomology.

5.2. Review: results of the previous papers [N2] and [N5]. Throughout this section, we denote the union of the fundamental groups of the boundaries of $S^3\backslash L$ by $\partial\pi_1(S^3\backslash L)$, for brevity. Let $m = \#L$, and choose a diagram D of L.

THEOREM 5.3 ([N2, Theorem 2.2]). *Let X be $M \times G$, as mentioned in* (2). *Let $\kappa : X \to M \rtimes G$ be a map which sends (m, g) to $(m - mg, g)$. Given an X-coloring $\mathscr{C}$ over f, consider a map* $\{\text{arcs of } D\} \to M \rtimes G$ *which assigns α to $\kappa(\mathscr{C}(\alpha))$. Then, this assignment yields isomorphisms*

$$\mathrm{Col}_X(D_f) \cong Z^1(\pi_1(S^3\backslash L), \partial\pi_1(S^3\backslash L); M),$$

$$\mathrm{Col}_X^{\mathrm{red}}(D_f) \cong H^1(\pi_1(S^3\backslash L), \partial\pi_1(S^3\backslash L); M).$$

Next, we explain Theorem 5.4. Choose a relative 1-cocycle $\tilde{f} : \pi_1(S^3\backslash L) \to M \rtimes \pi_1(S^3\backslash L)$ with $y_1, \dots, y_m$. We consider the subgroup K_ℓ defined by

$$\{(y_\ell - y_\ell \mathfrak{m}_\ell^a \mathfrak{l}_\ell^b, \mathfrak{m}_\ell^a \mathfrak{l}_\ell^b) \in M \rtimes \pi_1(S^3\backslash L) \mid a, b \in \mathbb{Z}^2\}.$$

Given a G-invariant trilinear map $\psi : M^3 \to A$, consider the map

$$\theta_\ell : (M \rtimes \pi_1(S^3\backslash L))^3 \to A;$$

$$((a, g), (b, h), (c, k)) \mapsto \psi((a + y_\ell - y_\ell g) \cdot hk, (b + y_\ell - y_\ell h) \cdot k, \quad c + y_\ell - y_\ell k). \tag{23}$$

Then, we can easily check that each θ_ℓ is a 3-cocycle in $C^3(M \rtimes \pi_1(S^3\backslash L); A)$. The collection $\Psi := (\theta_1, \dots, \theta_{\#L})$ represents a relative 3-cocycle in $D^3(M \rtimes \pi_1(S^3\backslash L), \mathscr{K}; A)$.

PROPOSITION 5.4 ([N5, Proposition 6.7]). *Under the notation above, fix a shadow coloring $\mathscr{S}_{\tilde{f}}$ corresponding the relative* 1*-cocycle $(\tilde{f}, y_1, \dots, y_{\#L})$.*

If L is either a hyperbolic link or a prime knot which is neither a cable knot nor a torus knot, as in Theorem 2.5*, then the diagonal restriction of $\mathscr{T}_\psi$ is equal to the pairing of the* 3*-class $[E_L, \partial E_L]$ and the above* 3*-cocycle Ψ. Namely,*

$$\mathscr{T}_\psi(\mathscr{S}_{\tilde{f}}, \mathscr{S}_{\tilde{f}}, \mathscr{S}_{\tilde{f}}) = \psi\langle \Psi, \tilde{f}_*[E_L, \partial E_L]\rangle. \tag{24}$$

If L is the (m, n)-torus knot, the same equality (24) *holds modulo mn.*

5.3. Proof of Theorem 2.5; trilinear pairing.

Proof (Proof of Theorem 2.5). First, we observe (25) below. Consider a 0-cochain $\vec{y} := (y_1, \dots, y_{\#L}) \in D^0(M \rtimes \pi_1(S^3\backslash L), M)$. Then, $\tilde{f} - \partial^0\vec{y}$ is represented by another 1-cocycle

$$\mathscr{C}' := ((\tilde{f} - \bar{y}_1, \dots, \tilde{f} - \bar{y}_{\#L}), (0, \dots, 0)) \in D^1(M \rtimes \pi_1(S^3\backslash L), M),$$

where $\bar{y}_\ell$ denotes the map $\pi_1(S^3\backslash L) \to M$ which takes g to $y_1 - y_1 g$. The 3-cocycle Ψ explained in (23) is equal to the cup product $\mathscr{C}' \smile \mathscr{C}' \smile \mathscr{C}'$, by definition. Hence, Proposition 5.4 implies

$$\begin{aligned}\mathscr{T}_\psi(\mathscr{S}_{\tilde{f}}, \mathscr{S}_{\tilde{f}}, \mathscr{S}_{\tilde{f}}) &= \psi\langle \mathscr{C}' \smile \mathscr{C}' \smile \mathscr{C}', [E_L, \partial E_L]\rangle \\ &= \psi\langle \mathscr{C} \smile \mathscr{C} \smile \mathscr{C}, [E_L, \partial E_L]\rangle. \qquad (25)\end{aligned}$$

Finally, we will deal with the non-diagonal parts, and complete the proof. Here, we define M to be the direct product $M_1 \times M_2 \times M_3$, and consider the j-th inclusion

$$\iota_j : M_j \to M = M_1 \times M_2 \times M_3; \qquad x \mapsto (\delta_{1j}x, \delta_{2j}x, \delta_{3j}x).$$

Thus, we can decompose $\mathscr{S}_{\tilde{f}}$ as $(\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3) \in \mathrm{Col}_{X_1}(D_f) \times \mathrm{Col}_{X_2}(D_f) \times \mathrm{Col}_{X_3}(D_f)$ componentwise. In addition, we define a G-invariant trilinear form

$$\bar{\psi} : M \times M \times M \to A; \qquad ((a,b,c),(d,e,f),(g,h,i)) \mapsto \psi(a,e,f).$$

Then, the transformation of the coefficients $\iota_1 \times \iota_2 \times \iota_3$ yields a diagram

$$\begin{array}{ccccc}\prod_{i=1}^3 H^1(E_L, \partial E_L; M_i) & \xrightarrow{\smile} & H^3(E_L, \partial E_L; M_1 \times M_2 \times M_3) & \xrightarrow{\psi\circ\langle \bullet, [E_L, \partial E_L]\rangle} & A \\ \Big\downarrow{\scriptstyle (\iota_1\times\iota_2\times\iota_3)_*} & & \Big\downarrow{\scriptstyle (\iota_1\times\iota_2\times\iota_3)_*} & & \Big\| \\ H^1(E_L, \partial E_L; M) & \xrightarrow{\smile_\Delta} & H^3(E_L, \partial E_L; M \times M \times M) & \xrightarrow{\bar{\psi}\circ\langle \bullet, [E_L, \partial E_L]\rangle} & A.\end{array}$$

Here, the left bottom $\smile_\Delta$ is defined by $a \mapsto a \smile a \smile a$. Then, we can verify the commutativity directly from the definitions. By Proposition 5.4, the bottom arrow is equal to the left hand side in (24). Hence, the pullback to $\prod_{i=1}^3 H^1(E_L, \partial E_L; M_i)$ is equal to the trilinear $\mathscr{T}_\psi$ as desired.

Proof (Proof of Theorem 2.6). Let L be the (m,n)-torus knot. According to the latter part in Theorem 5.4, we need discussions modulo mn. Then, the proof runs well in the same manner.

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