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Eigenvalue estimates for submanifolds in Hadamard manifolds and product manifolds $N \times \mathbb{R}$

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Abstract. In this paper, we investigate submanifolds with locally bounded mean curvature in Hadamard manifolds, product manifolds $N \times \mathbb{R}$, submanifolds with bounded φ-mean curvature in the hyperbolic space, and successfully give lower bounds for the weighted fundamental tone and the first eigenvalue of the p-Laplacian.

1. Introduction

Let (M, g) be an n-dimensional $(n \geq 2)$ smooth Riemannian manifold with the Riemannian metric g, the gradient operator ∇ and the Laplacian $\Delta = \operatorname{div} \circ \nabla$. For an open bounded connected domain $\Omega \subset M$, the classical Dirichlet eigenvalue problem on Ω is actually to find possible real numbers λ such that the boundary value problem (BVP for short)

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

has a nontrivial solution u. The desired real numbers λ are called *eigenvalues* of Δ, and the space of solutions of each λ is called its eigenspace which is a vector space. It is well known that for the BVP (1), the self-adjoint operator Δ only has the discrete spectrum whose elements (i.e., eigenvalues) can be listed increasingly as follows

$$0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \cdots \uparrow \infty,$$

and each associated eigenspace has finite dimension. λ_i $(i \geq 1)$ is called the ith Dirichlet eigenvalue of Δ. By *domain monotonicity of eigenvalues with*

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vanishing Dirichlet data (cf. [4, pp. 17–18]), we know that $\lambda_1(\Omega_1) \leq \lambda_1(\Omega_2)$ if $\Omega_1 \supset \Omega_2$.

For a domain $\Omega \subseteq M$ (with or without boundary $\partial\Omega$), one can define *the fundamental tone* $\lambda_1^*(\Omega)$ of Ω as

$$\lambda_1^*(\Omega) := \inf\left\{ \frac{\int_\Omega \|\nabla f\|^2 dv}{\int_\Omega f^2 \, dv} \,\middle|\, f \in W_0^{1,2}(\Omega), f \neq 0 \right\},$$

where $W_0^{1,2}(\Omega)$ is the completion of the set $C_0^\infty(\Omega)$ of smooth functions compactly supported on Ω under the Sobolev norm $\|u\|_{1,2} = \{\int_\Omega (|u|^2 + \|\nabla u\|^2) dv\}^{1/2}$, with dv the Riemannian volume element with respect to the metric g. In what follows, without specification, $\|\cdot\|$ *denotes the norm of some prescribed vector field*, and, *for the sake of simplicity, the measure dv will be omitted from integrals*. If Ω is unbounded, then the fundamental tone $\lambda_1^*(\Omega)$ coincides with the infimum $\inf(\Sigma)$ of the spectrum $\Sigma \subseteq [0, +\infty)$ of the unique self-adjoint extension of the Laplacian Δ acting on $C_0^\infty(\Omega)$, which is also denoted by Δ. If Ω has compact closure and piecewise smooth boundary $\partial\Omega$ (maybe nonempty), $\lambda_1^*(\Omega)$ equals the first closed eigenvalue (if $\partial\Omega = \emptyset$) or the first Dirichlet eigenvalue (if $\partial\Omega \neq \emptyset$) $\lambda_1(\Omega)$ of Δ. If $\Omega_1 \subset \Omega_2$ are bounded domains, then $\lambda_1^*(\Omega_1) \geq \lambda_1^*(\Omega_2) \geq 0$.

From the above introduction, we know that *for a bounded domain Ω with boundary, the degree of smoothness of the boundary $\partial\Omega$ decides the fundamental tone $\lambda_1^*(\Omega)$ would degenerate into the first Dirichlet eigenvalue $\lambda_1(\Omega)$ of the Laplacian or not*.

Let $B_M(q, \ell)$ be a geodesic ball, with center q and radius ℓ, on a complete noncompact Riemannian manifold M. By the monotonicity of the first Dirichlet eigenvalue λ_1 or the fundamental tone λ_1^*, one can define a limit $\lambda_1(M)$ by

$$\lambda_1(M) := \lim_{\ell\to\infty} \lambda_1(B_M(q,\ell)) = \lim_{\ell\to\infty} \lambda_1^*(B_M(q,\ell)),$$

which is independent of the choice of the center q. Clearly, $\lambda_1(M) \geq 0$. Schoen and Yau [18, p. 106] suggested that it is an important question to find conditions which will imply $\lambda_1(M) > 0$. Speaking in other words, manifolds with $\lambda_1(M) > 0$ might have some special geometric properties. There are many interesting results supporting this. For instance, Mckean [17] showed that for an n-dimensional complete noncompact, simply connected Riemannian manifold M with sectional curvature $K_M \leq -a^2 < 0$, $\lambda_1(M) \geq \frac{(n-1)^2 a^2}{4} > 0$, and moreover, $\lambda_1(\mathbb{H}^n(-a^2)) = \frac{(n-1)^2 a^2}{4}$ with $\mathbb{H}^n(-a^2)$ the n-dimensional hyperbolic space of sectional curvature $-a^2$. Grigor'yan [11] showed that if $\lambda_1(M) > 0$, then M is non-parabolic, i.e., there exists a non-constant bounded subharmonic function on M. Cheung and Leung [6] proved that if M is an n-dimensional complete minimal submanifold in the hyperbolic m-space $\mathbb{H}^m(-1)$, then $\lambda_1(M) \geq \frac{(n-1)^2}{4} > 0$, and moreover, M is non-parabolic. They also showed that if furthermore M has at least two ends, then there exists a non-constant bounded harmonic function on M with finite Dirichlet energy.

Consider the BVP

$$\begin{cases} \Delta_\varphi u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset M$ is an open bounded connected domain in a given Riemannian manifold M, $\Delta_\varphi u := \Delta u - \langle \nabla u, \nabla\varphi \rangle$ is the weighted Laplacian (also called the drifting Laplacian) on M, and φ is a real-valued smooth function on M. Similar to the BVP (1), Δ_φ in the BVP (2) only has the discrete spectrum and all the eigenvalues in the discrete spectrum can be listed increasingly. By Rayleigh's theorem and the max-min principle, it is easy to know that the first Dirichlet eigenvalue $\lambda_{1,\varphi}(\Omega)$ of Δ_φ on Ω can be characterized by

$$\lambda_{1,\varphi}(\Omega) = \inf\left\{ \frac{\int_\Omega \|\nabla f\|^2 e^{-\varphi}}{\int_\Omega f^2 e^{-\varphi}} \,\middle|\, f \in W_0^{1,2}(\Omega), f \neq 0 \right\}.$$

Similar to the case of the Laplacian, for a (bounded or unbounded) domain $\Omega \subseteq M$ (with or without boundary $\partial\Omega$), one can define *the weighted fundamental tone* $\lambda^*_{1,\varphi}(\Omega)$ of Ω as

$$\lambda^*_{1,\varphi}(\Omega) := \inf\left\{ \frac{\int_\Omega \|\nabla f\|^2 e^{-\varphi}}{\int_\Omega f^2 e^{-\varphi}} \,\middle|\, f \in W_0^{1,2}(\Omega), f \neq 0 \right\},$$

and it is not difficult to get that $\lambda^*_{1,\varphi}(\Omega) = \lambda_{1,\varphi}(\Omega)$ if Ω has compact closure and its boundary $\partial\Omega$ is piecewise smooth.

Domain monotonicity of eigenvalues with vanishing Dirichlet data also holds for the first Dirichlet eigenvalue of Δ_φ (see, e.g., [8, Lemma 1.5]). This implies that for a complete noncompact Riemannian manifold M, one can define the limit

$$\lambda_{1,\varphi}(M) := \lim_{\ell\to\infty} \lambda_{1,\varphi}(B_M(q,\ell)) = \lim_{\ell\to\infty} \lambda^*_{1,\varphi}(B_M(q,\ell)),$$

which is independent of the choice of the point q and can be seen as a generalization of $\lambda_1(M)$. Clearly, $\lambda_{1,\varphi}(M) \geq 0$ and if $\varphi = const.$, then $\lambda_{1,\varphi}(M) = \lambda_1(M)$. Based on Schoen-Yau's suggestion mentioned before, it is natural to ask:

Question 1. *For a given complete noncompact Riemannian manifold M, under what conditions, $\lambda_{1,\varphi}(M) > 0$?*

For an n-dimensional ($n \geq 2$) complete noncompact submanifold of a hyperbolic space whose norm of the mean curvature vector $\|H\|$ satisfies $\|H\| \leq \alpha < n-1$, Du and Mao [8, Theorem 1.7] proved that if $\|\varphi\| \leq C$[1], then $\lambda_{1,\varphi}(M) \geq \frac{(n-1-\alpha-C)^2}{4}$, with equality attained when M is totally geodesic and $\varphi = const.$, which generalized Cheung-Leung's and Mckean's conclusions mentioned before.

Consider the BVP

$$\begin{cases} \Delta_p u + \lambda |u|^{p-2} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\Omega \subset M$ is an open bounded connected domain in a given Riemannian manifold M, $\Delta_p u := \operatorname{div}(\|\nabla u\|^{p-2} \nabla u)$ is the nonlinear p-Laplacian of u with $1 < p < \infty$. It is known that (3) has a positive weak solution, which is unique modulo the scaling, in $W_0^{1,p}(\Omega)$, the completion of the set $C_0^\infty(\Omega)$ of smooth functions compactly supported on Ω under the Sobolev norm $\|u\|_{1,p} = \{\int_\Omega (|u|^p + \|\nabla u\|^p)\}^{1/p}$, and the first Dirichlet eigenvalue $\lambda_{1,p}(\Omega)$ of the p-Laplacian in the eigenvalue problem (3) can be characterized by

$$\lambda_{1,p}(\Omega) = \inf\left\{ \frac{\int_\Omega \|\nabla f\|^p}{\int_\Omega |f|^p} \,\middle|\, f \in W_0^{1,p}(\Omega), f \neq 0 \right\}.$$

The (closed or Dirichlet) eigenvalue problem of the p-Laplacian has been studied by the first named author and some interesting conclusions have been obtained (see, e.g., [7, 8, 13, 14]). Domain monotonicity of eigenvalues with vanishing Dirichlet data also holds for the first Dirichlet eigenvalue of Δ_p (see, e.g., [8, Lemma 1.1]). This implies that for a complete noncompact Riemannian manifold M, one can define the limit

$$\lambda_{1,p}(M) := \lim_{\ell \to \infty} \lambda_{1,p}(B_M(q, \ell)),$$

which is independent of the choice of the point q and can be seen as a generalization of $\lambda_1(M)$. Clearly, $\lambda_{1,p}(M) \geq 0$ and if $p = 2$, then $\lambda_{1,p}(M) = \lambda_1(M)$. Based on Schoen-Yau's suggestion mentioned before, it is natural to ask:

Question 2. *For a given complete noncompact Riemannian manifold M, under what conditions, $\lambda_{1,p}(M) > 0$?*

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[1] It is easy to know that the constant C satisfies $C < n - 1 - \alpha$, which is the potential assumption in [8, Theorem 1.7], since in the proof of [8, Theorem 1.7], the positive number ε is chosen to be $\varepsilon = (n - 1 - \alpha - C)/2$.

For an n-dimensional ($n \geq 2$) complete noncompact submanifold of a hyperbolic space whose norm of the mean curvature vector $\|H\|$ satisfies $\|H\| \leq \alpha < n-1$, Du and Mao [8, Theorem 1.3] proved $\lambda_{1,p}(M) \geq \left(\frac{n-1-\alpha}{p}\right)^p > 0$, with equality attained when M is totally geodesic and $p = 2$, which generalized Cheung-Leung's and Mckean's conclusions mentioned before.

The purpose of this paper is trying to positively answer Questions 1 and 2 *further*. In fact, we have obtained the following facts:

- By introducing a quantity $c(\Omega)$ for a domain Ω with compact closure (see Definition 1), Bessa-Montenegro type lower bounds for the weighted fundamental tone $\lambda^*_{1,\varphi}(\Omega)$ and the first eigenvalue $\lambda_{1,p}(\Omega)$ of the p-Laplacian can be obtained—see Lemma 1. By applying the Hessian comparison theorem, domain monotonicity of eigenvalues with vanishing Dirichlet data for $\lambda^*_{1,\varphi}(\cdot)$ and $\lambda_{1,p}(\cdot)$, Bessa-Montenegro type lower bounds would give us Mckean-type lower bounds for Hadamard manifolds with strictly negative sectional curvature—see Lemma 2.
- Let $\phi : M \to Q$ be an isometric immersion from n-dimensional ($n \geq 2$) Riemannian manifold to an m-dimensional Riemannian manifold, and moreover, M has locally bounded mean curvature (see Definition 2). For any connected component Ω of $\phi^{-1}(\overline{B_Q(q,r)})$ with $q \in Q \backslash \phi(M)$, and $r > 0$, under different assumptions on sectional curvatures, some strictly positive lower bounds have been obtained for the weighted fundamental tone $\lambda^*_{1,\varphi}(\Omega)$ (no matter Ω is bounded or unbounded) and the first eigenvalue $\lambda_{1,p}(\Omega)$ of the p-Laplacian (in this case, Ω is bounded and has piecewise smooth boundary)—see Theorem 2. As a direct consequence, if furthermore M is noncompact with bounded mean curvature (stronger than the *locally* bounded mean curvature assumption) and the sectional curvature of Q is bounded from above by some strictly negative constant, then $\lambda_{1,\varphi}(M)$ and $\lambda_{1,p}(M)$ have strictly positive lower bounds—see Corollary 4.
- Recently, because of the discovery of many interesting examples of minimal surfaces in product spaces $N \times \mathbb{R}$ (see, e.g., [15, 16]), the study of this kind of spaces has attracted geometers' attention. Based on this, we investigate submanifolds Ω, with locally bounded mean curvature, of $N \times \mathbb{R}$ and would like to know "*under what conditions,* $\lambda^*_{1,\varphi}(\Omega) > 0$ *and* $\lambda_{1,p}(\Omega) > 0$*?*". A positive answer has been given—see Theorem 3 for details.
- For an n-dimensional ($n \geq 2$) complete non-compact φ-minimal submanifold M of the weighted manifold $(\mathbb{H}^m(-1), e^{-\varphi}\, dv)$, where $\mathbb{H}^m(-1)$ is the hyperbolic m-space with sectional curvature -1, φ is a real-valued smooth function on $\mathbb{H}^m(-1)$ and dv is the volume element, a strictly

positive lower bound has been obtained for the first eigenvalue $\lambda_{1,p}(M)$ for the p-Laplacian on M—see Theorem 4 for details.

- Interesting *new* lower bounds for the first Dirichlet eigenvalues of the weighted Laplacian and the p-Laplacian on geodesic balls of complete Riemannian manifolds have been given—see Theorem 5 for details.

2. Bessa-Montenegro type and Mckean-type lower bounds for the weighted fundamental tone and the first eigenvalue of the p-Laplacian

By using a notion introduced in [1], we can give lower bounds for the weighted fundamental tone for arbitrary bounded domains, and the lowest eigenvalue for the Dirichlet eigenvalue problem of the weighted Laplacian and the p-Laplacian on normal domains.

DEFINITION 1 ([1]). Let $\Omega \subset M$ be a domain with compact closure in a C^∞ Riemannian manifold M. Let $\mathscr{X}(\Omega)$ be the set of all smooth vector fields X on Ω with $\|X\|_\infty := \sup_\Omega \|X\| < \infty$ and $\inf \operatorname{div} X > 0$ with div the divergence operator on M. Define $c(\Omega)$ by

$$c(\Omega) := \sup\left\{\frac{\inf \operatorname{div} X}{\|X\|_\infty} : X \in \mathscr{X}(\Omega)\right\}. \tag{4}$$

REMARK 1. As shown in [1, Remark 2.2], it is easy to get that $\mathscr{X}(\Omega)$ is not empty. This is because the boundary value problem (BVP for short)

$$\begin{cases} \Delta u = 1, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

always has a solution on a bounded domain $\Omega \subset M$, and then at least one can choose $X = \nabla u$, the gradient of u, which implies that $\operatorname{div}(X) = 1$ and $\|X\|_\infty < \infty$.

Now, we can prove the following.

LEMMA 1. *Let $\Omega \subset M$ be a domain with compact closure and nonempty boundary (i.e., $\partial\Omega \neq \emptyset$) in a Riemannian manifold M. Then we have*

$$\lambda_{1,\varphi}^*(\Omega) \geq \frac{(c(\Omega) - c^+)^2}{4} > 0$$

provided $\|\nabla\varphi\| \leq c^+ < c(\Omega)$, where c^+ is the supremum of the norm of the gradient of φ and is strictly less than $c(\Omega)$, and $c(\Omega)$ is given by (4). *Moreover, if furthermore the boundary $\partial\Omega$ is piecewise smooth, then we have*

$$\lambda_{1,p}(\Omega) \geq \left(\frac{c(\Omega)}{p}\right)^p > 0.$$

PROOF. Taking $f \in C_0^\infty(\Omega)$, the set of all smooth functions compactly supported on Ω, and $X \in \mathscr{X}(\Omega)$. By a direct calculation, we have

$$\begin{aligned}\operatorname{div}(|f|^p X) &= \langle \nabla |f|^p, X\rangle + |f|^p \operatorname{div} X \\ &\geq -p|f|^{p-1}\|\nabla f\| \sup\|X\| + \inf \operatorname{div} X \cdot |f|^p. \qquad (5)\end{aligned}$$

By Young's inequality, one can obtain

$$|f|^{p-1}\|\nabla f\| = \varepsilon |f|^{p-1} \cdot \frac{\|\nabla f\|}{\varepsilon} \leq \frac{\left(\frac{\|\nabla f\|}{\varepsilon}\right)^p}{p} + \frac{(\varepsilon |f|^{p-1})^{p/(p-1)}}{\frac{p}{p-1}},$$

where $\varepsilon > 0$ is a parameter determined later. Substituting the above inequality into (5) yields

$$\operatorname{div}(|f|^p X) \geq -p \sup\|X\| \left[\frac{\left(\frac{\|\nabla f\|}{\varepsilon}\right)^p}{p} + \frac{(\varepsilon |f|^{p-1})^{p/(p-1)}}{\frac{p}{p-1}}\right] + \inf \operatorname{div} X \cdot |f|^p. \quad (6)$$

Choosing

$$\varepsilon = \left(\frac{\inf \operatorname{div} X}{p \sup\|X\|}\right)^{(p-1)/p},$$

in (6), integrating both sides of (6) over Ω and using the divergence theorem, we have

$$\int_\Omega \|\nabla f\|^p \geq \left(\frac{\inf \operatorname{div} X}{p \sup\|X\|}\right)^p \int_\Omega |f|^p, \qquad (7)$$

which implies

$$\lambda_{1,p}(\Omega) \geq \left(\frac{c(\Omega)}{p}\right)^p$$

by taking the supremum over all vector fields $X \in \mathscr{X}(\Omega)$ to the RHS of (7).

If $\|\nabla\varphi\| \leq c^+ < c(\Omega)$ with $c^+ \geq 0$ the supremum of $\|\nabla\varphi\|$, then we have

$$\begin{aligned}\operatorname{div}(f^2 X e^{-\varphi}) &= e^{-\varphi}\langle \nabla f^2, X\rangle + f^2 e^{-\varphi} \operatorname{div} X - f^2 e^{-\varphi}\langle \nabla\varphi, X\rangle \\ &\geq e^{-\varphi}[-2|f| \cdot \|\nabla f\| \cdot \sup\|X\| + f^2 \inf \operatorname{div} X - f^2 c^+ \sup\|X\|] \\ &\geq e^{-\varphi}\left[\left(-\varepsilon f^2 - \frac{\|\nabla f\|^2}{\varepsilon}\right) \sup\|X\| + f^2 \inf \operatorname{div} X \right. \\ &\qquad \left. - f^2 c^+ \sup\|X\|\right], \qquad (8)\end{aligned}$$

where $\varepsilon > 0$ is a parameter determined later. Integrating both sides of (8) and using the divergence theorem, we have

$$\int_\Omega \|\nabla f\|^2 e^{-\varphi} \geq \frac{\varepsilon(\inf \operatorname{div} X - c^+ \sup\|X\| - \varepsilon \sup\|X\|)}{\sup\|X\|} \int_\Omega f^2 e^{-\varphi}. \tag{9}$$

On the other hand, since

$$\frac{\varepsilon(\inf \operatorname{div} X - c^+ \sup\|X\| - \varepsilon \sup\|X\|)}{\sup\|X\|} \leq \left(\frac{\frac{\inf \operatorname{div} X}{\sup\|X\|} - c^+}{2}\right)^2$$

with equality holds if and only if $\varepsilon = \frac{\inf \operatorname{div} X}{2 \sup\|X\|} - \frac{c^+}{2} > 0$, we can obtain

$$\lambda_{1,\varphi}(\Omega) \geq \frac{(c(\Omega) - c^+)^2}{4} > 0$$

by choosing $\varepsilon = \frac{\inf \operatorname{div} X}{2 \sup\|X\|} - \frac{c^+}{2}$ in (9) and by taking the supremum over all vector fields $X \in \mathscr{X}(\Omega)$. This completes the proof of Lemma 1.

REMARK 2. (1) Clearly, when $p = 2$ (or $\varphi = const.$), the nonlinear p-Laplacian (or the weighted Laplacian) degenerate into the Laplacian. Correspondingly, $\lambda_{1,p}(\Omega) = \lambda_1^*(\Omega)$ (or $\lambda_{1,\varphi}(\Omega) = \lambda_1^*(\Omega)$, $c^+ = 0$), and moreover, $\lambda_1^*(\Omega) \geq \left(\frac{c(\Omega)}{2}\right)^2$, which is the lower bound for $\lambda_1^*(\Omega)$ in [1, Lemma 2.3] given by Bessa and Montenegro. Based on this fact, we would like to use *Bessa-Montenegro type lower bounds* to call the lower bounds for the lowest Dirichlet eigenvalue (resp., the weighted fundamental tone) shown in Lemma 1. Besides, to prove Bessa-Montenegro type lower bounds here, we only need to consider vector fields smooth almost every in Ω such that $\int_\Omega \operatorname{div}(|f|^p X) = 0$ or $\int_\Omega \operatorname{div}(f^2 X e^{-\varphi}) = 0$ for all $f \in C_0^\infty(\Omega)$.

(2) It has been shown in [1, Remark 2.7] that $c(\Omega) \leq h(\Omega)$ with $h(\Omega) := \inf_{A \subset \Omega} \frac{\operatorname{vol}(\partial A)}{\operatorname{vol}(A)}$ the Cheeger's constant. However, in some cases, for instance, for balls in the Euclidean space or Hadamard manifolds, $c(\Omega) = h(\Omega)$. The advantage of defining $c(\Omega)$ is the computability of lower bounds for $\lambda_{1,p}(\Omega)$, $\lambda_{1,\varphi}(\Omega)$ via any lower bound for $c(\Omega)$, and this way can be applied to arbitrary domains. Besides, we can use Lemma 1 to derive *Mckean-type* lower bounds below—see Lemma 2 for details.

Applying Lemma 1, one can get the following conclusion directly.

COROLLARY 1. *Let $\Omega \subset M$ be a normal domain with compact closure in a smooth Riemannian manifold M. For the BVP*

$$\begin{cases} \Delta v = 1, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases}$$

we have

$$\lambda_{1,p}(\Omega) \geq \left(\frac{1}{p\|\nabla v\|_{\infty}}\right)^{p} > 0.$$

Besides,

$$\lambda_{1,\varphi}^{*}(\Omega) \geq \frac{\left(\frac{1}{\|\nabla v\|_{\infty}} - c^{+}\right)^{2}}{4} > 0$$

provided $\|\nabla\varphi\| \leq c^{+} < \frac{1}{\|\nabla v\|_{\infty}}$*, where* c^{+} *is the supremum of the norm of the gradient of* φ *and is strictly less than* $\frac{1}{\|\nabla v\|_{\infty}}$.

Corollary 2. *There are no smooth bounded vector fields* $X : M \to TM$ *with* $\inf_M \operatorname{div} X > 0$ *on complete noncompact manifolds* M *such that* $\lambda_{1,p}(M) = 0$*,* $\lambda_{1,\varphi}(M) = 0$*. In particular, there is no such vector field on* $\mathbb{R}^n$.

As an interesting application of Lemma 1, we can obtain Mckean-type lower bounds for the first eigenvalues of the drifting Laplacian and the p-Laplacian on the prescribed Hadamard manifold. However, in order to prove that, we need to use *the Hessian comparison theorem* below.

Theorem 1 (Hessian comparison theorem). *Let* M *be a complete Riemannian manifold and* $x_0, x \in M$*. Let* $\gamma : [0, \rho(x)] \to M$ *be a minimizing geodesic joining* x_0 *and* x*, where* $\rho(x)$ *is the distance function* $\operatorname{dist}_M(x_0, x)$*. Let* K *be the sectional curvature of* M *and* $\mu_i(\rho)$*,* $i = 0, 1$*, be functions defined by*

$$\mu_0(\rho) = \begin{cases} k_0 \coth(k_0\rho(x)), & \text{if } \inf_\gamma K = -k_0^2, \\ \frac{1}{\rho(x)}, & \text{if } \inf_\gamma K = 0, \\ k_0 \cot(k_0\rho(x)), & \text{if } \inf_\gamma K = k_0^2 \text{ and } \rho < \frac{\pi}{2k_0} \end{cases}$$

and

$$\mu_1(\rho) = \begin{cases} k_1 \coth(k_1\rho(x)), & \text{if } \sup_\gamma K = -k_1^2, \\ \frac{1}{\rho(x)}, & \text{if } \sup_\gamma K = 0, \\ k_1 \cot(k_1\rho(x)), & \text{if } \sup_\gamma K = k_1^2 \text{ and } \rho < \frac{\pi}{2k_1}. \end{cases}$$

Then the Hessians of ρ *and* ρ^2 *satisfy*

$$\mu_1(\rho(x)) \cdot \|X\|^2 \leq \operatorname{Hess}\rho(x)(X, X) \leq \mu_0(\rho(x)) \cdot \|X\|^2,$$

$$\operatorname{Hess}\rho(x)(\gamma', \gamma') = 0,$$

$$2\rho(x)\cdot\mu_1(\rho(x))\cdot\|X\|^2 \le \mathrm{Hess}\,\rho^2(x)(X,X) \le 2\rho(x)\cdot\mu_0(\rho(x))\cdot\|X\|^2,$$
$$\mathrm{Hess}\,\rho^2(x)(\gamma',\gamma') = 2,$$

where X is any vector in T_xM perpendicular to $\gamma'(\rho(x))$.

Hence, by applying Theorem 1, for the distance function $\rho(x)$ on an n-dimensional Riemannian manifold M, we can get

$$2(n-1)\rho(x)\mu_1(\rho(x)) + 2 \le \Delta\rho^2(x) \le 2(n-1)\rho(x)\mu_0(\rho(x)) + 2. \tag{10}$$

Lemma 2. *Let M be an n-dimensional $(n \ge 2)$ Hadamard manifold whose sectional curvature satisfies $K_M \le -a^2 < 0$, $a > 0$. Then we have*

$$\lambda_{1,p}(M) \ge \left[\frac{(n-1)\cdot a}{p}\right]^p > 0.$$

Moreover,

$$\lambda_{1,\varphi}(M) \ge \left[\frac{(n-1)\cdot a - c^+}{2}\right]^2 > 0$$

provided $\|\nabla\varphi\| \le c^+ < (n-1)a$, where c^+ is the supremum of the norm of the gradient of φ and is strictly less than $(n-1)a$.

Proof. Let $\rho : M \to \mathbb{R}$ be the distance function to a point $p \in M\backslash\Omega$ with Ω a normal domain in M, and let $X = \nabla\rho$. By (10), we have

$$\Delta\rho(x) = \mathrm{div}\, X \ge (n-1)\cdot a\cdot \coth(a\cdot\rho(x)) \ge (n-1)\cdot a.$$

By Lemma 1, it follows that

$$\lambda_{1,p}(\Omega) \ge \left[\frac{(n-1)\cdot a}{p}\right]^p$$

and

$$\lambda_{1,\varphi}(\Omega) = \lambda^*_{1,\varphi}(\Omega) \ge \left[\frac{(n-1)\cdot a - c^+}{2}\right]^2,$$

which, by [8, Lemma 1.1], implies the lower bounds for $\lambda_{1,p}(M)$, $\lambda_{1,\varphi}(M)$ in Lemma 2.

Remark 3. Clearly, when $p = 2$ (or $\varphi = const.$), the nonlinear p-Laplacian (or the weighted Laplacian) degenerate into the Laplacian. Correspondingly, $\lambda_{1,p}(M) = \lambda_1(M)$ (or $\lambda_{1,\varphi}(M) = \lambda_1(M)$, $c^+ = 0$), and moreover, $\lambda_1(M) \ge \frac{(n-1)^2a^2}{4} > 0$, which is exactly Mckean's lower bound shown in [17].

3. Eigenvalue estimates for submanifolds with locally bounded mean curvature in Hadamard manifolds

Let $\phi : M \to Q$ be an isometric immersion with M, Q complete Riemannian manifolds, $\dim(M) = n$, $n \geq 2$. Consider a smooth function $g : Q \to \mathbb{R}$ and the composition $f = g \circ \phi : M \to \mathbb{R}$. As before, let Δ be the Laplace operator on M. *However, because of the isometric immersion, for convenience, in this section, we can use* $\mathrm{grad}(\cdot)$ *to denote the gradient of a given function on* M *or its isometric image* $\phi(M) \subseteq Q$. Identify X with $d\phi(X)$, and then we can obtain that at $q \in M$,

$$\langle \mathrm{grad}\, f, X \rangle = df(X) = dg(X) = \langle \mathrm{grad}\, g, X \rangle$$

for every $X \in T_q M$. Therefore, it follows that

$$\mathrm{grad}\, g = \mathrm{grad}\, f + (\mathrm{grad}\, g)^{\perp},$$

with $(\mathrm{grad}\, g)^{\perp}$ perpendicular to $T_q M$. For $X, Y \in T_q M$, let $\alpha(q)(X, Y)$ and $\mathrm{Hess}\, f(q)(X, Y)$ be the second fundamental form of the immersion ϕ and the Hessian of f at $q \in M$, respectively. By the Gauss equation, we have

$$\mathrm{Hess}\, f(q)(X, Y) = \mathrm{Hess}\, g(\phi(q))(X, Y) + \langle \mathrm{grad}\, g, \alpha(X, Y) \rangle_{\phi(q)}. \tag{11}$$

Taking the trace in (11) w.r.t. an orthonormal basis $\{e_1, e_2 \dots e_n\}$ of $T_q M$, we can get

$$\Delta f(q) = \sum_{i=1}^{n} \mathrm{Hess}\, f(q)(e_i, e_i) = \sum_{i=1}^{n} \mathrm{Hess}\, g(\phi(q))(e_i, e_i) + \left\langle \mathrm{grad}\, g, \sum_{i=1}^{n} \alpha(e_i, e_i) \right\rangle. \tag{12}$$

See, e.g., [6, 8] for more generalized versions of the formulas (11) and (12) above.

We need the following notion.

Definition 2. An isometric immersion $\phi : M \to Q$ has locally bounded mean curvature H if for any $q \in Q$ and $r > 0$, the number $h(q, r) := \sup\{\|H(x)\|; x \in \phi(M) \cap B_Q(q, r)\}$ is finite, where, as before, $B_Q(q, r)$ denotes the geodesic ball, with center q and radius r, on Q.

By using Lemma 1, Theorem 1 and the locally bounded mean curvature assumption, we can prove the following.

THEOREM 2. *Let* $\phi : M \to Q$ *be an isometric immersion with locally bounded mean curvature and let* Ω *be any connected component of* $\phi^{-1}(\overline{B_Q(q,r)})$, *where* $q \in Q \backslash \phi(M)$, $r > 0$ *and* $\dim(M) = n$, $n \geq 2$. *Let* $\kappa(q,r) = \sup\{K_Q(x) \mid x \in B_Q(q,r)\}$, *where* $K_Q(x)$ *is the sectional curvature at* x. *Denote by* $\mathrm{inj}(q)$ *the injectivity radius of* Q *at the point* q. *Assume that* φ *is a real-valued smooth function on* M *with* $\|\mathrm{grad}\,\varphi\| \leq c^+$, *where* c^+ *is the supremum of the norm of the gradient of* φ. *Choosing* r *properly, we have the following estimates:*

(1) *If* $\kappa(q, \mathrm{inj}(q)) = k^2 < \infty$, $k > 0$, *choose*

$$r < \min\left\{\mathrm{inj}(q), \frac{\pi}{2k}, \cot^{-1}\left[\frac{h(q,\mathrm{inj}(q))}{(n-1)k}\right] \Big/ k\right\}.$$

Then we have

$$\lambda^*_{1,\varphi}(\Omega) \geq \left[\frac{(n-1)k\cot(kr) - h(q,r) - c^+}{2}\right]^2$$

provided $c^+ < (n-1)k\cot(kr) - h(q,r)$. *If furthermore the boundary* $\partial\Omega$ *is piecewise smooth, then we have*

$$\lambda_{1,p}(\Omega) \geq \left[\frac{(n-1)k\cot(kr) - h(q,r)}{p}\right]^p.$$

(2) *If* $\lim_{\ell \to \infty} \kappa(q,\ell) = \infty$, *let*

$$r(s) := \min\left\{\frac{\pi}{2\sqrt{\kappa(q,s)}}, \cot^{-1}\left[\frac{h(q,s)}{(n-1)\sqrt{\kappa(q,s)}}\right] \Big/ \sqrt{\kappa(q,s)}\right\}, \qquad s > 0.$$

Choose $r = \max_{s>0} r(s)$. *We have*

$$\lambda^*_{1,\varphi}(\Omega) \geq \left[\frac{(n-1)\sqrt{\kappa(q,s)}\cot(\sqrt{\kappa(q,s)}r) - h(q,r) - c^+}{2}\right]^2$$

provided $c^+ < (n-1)\sqrt{\kappa(q,s)}\cot(\sqrt{\kappa(q,s)}r) - h(q,r)$. *If furthermore the boundary* $\partial\Omega$ *is piecewise smooth, then we have*

$$\lambda_{1,p}(\Omega) \geq \left[\frac{(n-1)\sqrt{\kappa(q,s)}\cot(\sqrt{\kappa(q,s)}r) - h(q,r)}{p}\right]^p.$$

(3) *If* $\kappa(q,\mathrm{inj}(q)) = 0$, *choose* $r < \min\left\{\mathrm{inj}(q), \frac{n}{h(q,\mathrm{inj}(q))}\right\}$. *Assume that* $\frac{n}{h(q,\mathrm{inj}(q))} = \infty$ *if* $h(q,\mathrm{inj}(q)) = 0$. *Then we have*

$$\lambda^*_{1,\varphi}(\Omega) \geq \left[\frac{\frac{n}{r} - h(q,r) - c^+}{2}\right]^2$$

provided $c^+ < \frac{n}{r} - h(q,r)$. *If furthermore* Ω *is bounded and its boundary* $\partial\Omega$ *is piecewise smooth, then we have*

$$\lambda_{1,p}(\Omega) \geq \left[\frac{\frac{n}{r} - h(q,r)}{p}\right]^p.$$

(4) *If* $\kappa(q, \mathrm{inj}(q)) = -k^2 < \infty$, $k > 0$, *and* $h(q, \mathrm{inj}(q)) < (n-1)k$, *choose* $r < \mathrm{inj}(q)$. *Then*

$$\lambda^*_{1,\varphi}(\Omega) \geq \left[\frac{(n-1)k - h(q,r) - c^+}{2}\right]^2$$

provided $c^+ < (n-1)k - h(q,r)$. *If furthermore* Ω *is bounded and its boundary* $\partial\Omega$ *is piecewise smooth, then we have*

$$\lambda_{1,p}(\Omega) \geq \left[\frac{(n-1)k - h(q,r)}{p}\right]^p.$$

(5) *If* $\kappa(q, \mathrm{inj}(q)) = -k^2 < \infty$, $k > 0$, *and* $h(q, \mathrm{inj}(q)) \geq (n-1)k$, *choose*

$$r < \min\left\{\mathrm{inj}(q), \coth^{-1}\left[\frac{h(q, \mathrm{inj}(q))}{(n-1)k}\right]\Big/ k\right\}.$$

Then we have

$$\lambda^*_{1,\varphi}(\Omega) \geq \left[\frac{(n-1)k\coth(kr) - h(q,r) - c^+}{2}\right]^2$$

provided $c^+ < (n-1)k\coth(kr) - h(q,r)$. *If furthermore the boundary* $\partial\Omega$ *is piecewise smooth, then we have*

$$\lambda_{1,p}(\Omega) \geq \left[\frac{(n-1)k\coth(kr) - h(q,r)}{p}\right]^p.$$

In (2), *since* $r(s) > 0$ *for small* s, $r > 0$. *In* (3)–(5), *because of the non-positivity assumption on* $\kappa(q, \mathrm{inj}(q))$, *the radius* r *is not necessary to be finite, which implies that the connected component* Ω *of* $\phi^{-1}(\overline{B_Q(q,r)})$ *may be unbounded as* $r \to \infty$. *Besides, in* (4), *one can have a slight better estimate as follows*

$$\lambda^*_{1,\varphi}(\Omega) \geq \left[\frac{(n-1)k + \frac{1}{r} - h(q,r) - c^+}{2}\right]^2$$

provided $c^+ < (n-1)k + \frac{1}{r} - h(q,r)$, *by choosing* $X = \mathrm{grad}(\rho^2 \circ \phi)$ *in the proof below.*

PROOF. Similar to the proof of [1, Theorem 4.3]. Define two functions as follows

$$f_i = \rho^i \circ \phi : M \to \mathbb{R}, \qquad i = 1, 2,$$

where $\rho(x) = \mathrm{dist}_Q(q, x)$ is the distance function on Q. Clearly, f_1, f_2 are smooth functions on $\phi^{-1}(B_Q(q, \mathrm{inj}(q)))$. Let Ω be a connected component of $\phi^{-1}(\overline{B_Q(q, r)}) \subseteq \phi^{-1}(B_Q(q, \mathrm{inj}(q)))$, and let $X_i = \mathrm{grad}\, f_i$, $i = 1, 2$, on Ω. By (12), we have

$$\mathrm{div}\, X_i(x) = \Delta f_i(x) = \sum_{j=1}^{n-1} \mathrm{Hess}\, \rho^i(\phi(x))(e_j, e_j) + \langle \mathrm{grad}\, \rho^i, H \rangle_{\phi(x)},$$

with $\{e_1, e_2, \ldots, e_n\}$ an orthonormal basis of $T_x M$, where $e_n = \mathrm{grad}\, \rho(x)$. Applying Theorem 1 directly, one can obtain

- if $\kappa(q, \mathrm{inj}(q)) = k^2 < \infty$, $k > 0$, then $\mathrm{div}\, X_1 \geq (n-1)k \cot(kr) - h(q, r) > 0$;
- if $\kappa(q, \mathrm{inj}(q)) = 0$, then $\mathrm{div}\, X_2 \geq 2n - 2rh(q, r) > 0$;
- if $\kappa(q, \mathrm{inj}(q)) = -k^2 < \infty$, $k > 0$, then $\mathrm{div}\, X_1 \geq (n-1)k \coth(kr) - h(q, r) > 0$.

Together with the fact that $\|X_1\| = 1$ and $\|X_2\| = 2r$, estimates in Theorem 2 can be obtained by applying Lemma 1 directly.

REMARK 4. Clearly, when $\varphi = const.$ (or $p = 2$, Ω is bounded), our estimates here are exactly those in [1, Theorem 4.3].

Applying directly Theorem 2, we can obtain

COROLLARY 3. *Let $\phi : M \to \mathbb{R}^m$ be an isometric minimal immersion of an n-dimensional $(n \geq 2)$ complete submanifold. Assume that $\phi(M) \subset B_{\mathbb{R}^m}(o, r)$, then $\lambda_{1,p}(M) \geq \left(\frac{n}{pr}\right)^p$.*

Using a similar proof to that of [1, Corollary 4.4] and applying directly Theorem 2, [11, Proposition 10.1], [18, Theorem A.3], we can get the following.

COROLLARY 4. *Let $\phi : M \to Q$ be an isometric immersion with bounded mean curvature $\|H\| \leq \alpha < (n-1)a$, where M is an n-dimensional complete non-compact Riemannian manifold and Q is an m-dimensional complete simply connected Riemannian manifold with sectional curvature K_Q satisfying $K_Q \leq -a^2 < 0$ for some constant $a > 0$. Assume that φ is a real-valued smooth function on M with $\|\mathrm{grad}\, \varphi\| \leq c^+$, where c^+ is the supremum of the norm of the gradient of φ. Then we have the following estimates*

$$\lambda_{1,\varphi}(M) \geq \left[\frac{(n-1)a - \alpha - c^+}{2}\right]^2 > 0 \qquad (\textit{provided } c^+ < (n-1)a - \alpha)$$

and

$$\lambda_{1,p}(M) \geq \left[\frac{(n-1)a-\alpha}{p}\right]^p > 0.$$

In particular, there exist entire Green's functions on M. If furthermore M is minimal, then M is non-parabolic.

REMARK 5. Corollary 4 gives a positive answer to Questions 1 and 2 proposed in Section 1, i.e., finding conditions such that $\lambda_{1,\varphi}(M) > 0$, $\lambda_{1,p}(M) > 0$ for a complete noncompact manifold M, and also shows interesting geometric conclusions, i.e., the existence of Green's functions and the non-parabolic property. Besides, if $Q = \mathbb{H}^m(-1)$ which implies $a = 1$, then our lower bounds here are exactly those in [8, Theorems 1.3 and 1.7].

4. Eigenvalue estimates for submanifolds with locally bounded mean curvature in product manifolds $N \times \mathbb{R}$

Let $\phi : M \to N \times \mathbb{R}$ be an isometric immersion from an n-dimensional complete Riemannian manifold to the product space $N \times \mathbb{R}$ with N an m-dimensional complete Riemannian manifold. Since ϕ is an isometric immersion, we have formulas (11), (12) with $Q = N \times \mathbb{R}$. Besides, for convenience, *we can use* $\operatorname{grad}(\cdot)$ *to denote the gradient of a given function on M or its isometric image $\phi(M) \subseteq N \times \mathbb{R}$.* In this section, we would like to estimate from below the first fundamental tone $\lambda^*_{1,\varphi}(\Omega)$ of Ω (with $\Omega \subseteq M$) and the first eigenvalue $\lambda_{1,p}(\Omega)$ of the p-Laplacian on Ω (with $\Omega \subset M$ a domain with compact closure and piecewise smooth boundary). However, before that, we need the following notion, which is stronger than the one in Definition 2.

DEFINITION 3 ([3]). An isometric immersion $\phi : M \to N \times \mathbb{R}$ has locally bounded mean curvature H if for any $q \in N$ and $r > 0$, the number $h(q,r) := \sup\{\|H(x)\|; x \in \phi(M) \cap (B_N(q,r) \times \mathbb{R})\}$ is finite, where $B_N(q,r)$ denotes the geodesic ball, with center q and radius r, on N.

We also need the following conclusion, which is an extension of [2, Theorem 1.7].

LEMMA 3. *Let $\mathscr{W}^{1,1}(M)$ be the Sobolev space of all vector fields $X \in L^1_{loc}(M)$ possessing weak divergence*[2] $\operatorname{div} X$ *on a Riemannian manifold M.*

[2] For a Riemannian manifold M, a function $g \in L^1_{loc}(M)$ is a weak divergence of X if $\int_M g\psi = -\int_M \langle \operatorname{grad} \psi, X\rangle$, $\forall \psi \in C_0^\infty(M)$. There exists at most one $g \in L^1_{loc}(M)$ for a given vector field $X \in L^1_{loc}(M)$ and we can write $g = \operatorname{div} X$. Clearly, for a C^1 vector field X, its classical divergence coincides with the weak divergence $\operatorname{div} X$.

Assume that φ is a real-valued smooth function on M with $\|\mathrm{grad}\,\varphi\| \le c^+$, where c^+ is the supremum of the norm of the gradient of φ. Then the weighted fundamental tone $\lambda^*_{1,\varphi}(M)$ of M satisfies

$$\lambda^*_{1,\varphi}(M) \ge \sup_{\mathscr{W}^{1,1}(M)} \left\{ \inf_M (\mathrm{div}\, X - \|X\|^2 - c^+\|X\|) \right\}. \tag{13}$$

If furthermore M is complete, then the first eigenvalue $\lambda_{1,p}(M)$ of the p-Laplacian satisfies

$$\lambda_{1,p}(M) \ge \sup_{\mathscr{W}^{1,1}(M)} \left\{ \inf_M [\mathrm{div}\, X - (p-1)\|X\|^{p/(p-1)}] \right\}. \tag{14}$$

PROOF. Let $X \in L^1_{loc}(M)$ and $f \in C_0^\infty(M)$. Clearly, we have $\int_M \mathrm{div}(f^2Xe^{-\varphi}) = 0$ and $\int_M \mathrm{div}(|f|^pX) = 0$. By a direct computation, it follows that

$$\begin{aligned}
0 &= \int_M \mathrm{div}(f^2Xe^{-\varphi}) \\
&= \int_M f^2\, \mathrm{div}\, X \cdot e^{-\varphi} + \int_M \langle \mathrm{grad}\, f^2, X\rangle e^{-\varphi} - \int_M f^2 \langle \mathrm{grad}\,\varphi, X\rangle e^{-\varphi} \\
&\ge \int_M f^2\, \mathrm{div}\, X \cdot e^{-\varphi} - 2\int_M |f| \cdot \|X\| \cdot \|\mathrm{grad}\, f\| e^{-\varphi} - c^+ \int_M \|X\| f^2 e^{-\varphi} \\
&\ge \int_M f^2\, \mathrm{div}\, X \cdot e^{-\varphi} - \int_M [f^2 \cdot \|X\|^2 + \|\mathrm{grad}\, f\|^2] e^{-\varphi} - c^+ \int_M \|X\| f^2 e^{-\varphi} \\
&= \int_M (\mathrm{div}\, X - \|X\|^2 - c^+\|X\|) f^2 e^{-\varphi} - \int_M \|\mathrm{grad}\, f\|^2 e^{-\varphi} \\
&\ge \inf_M (\mathrm{div}\, X - \|X\|^2 - c^+\|X\|) \int_M f^2 e^{-\varphi} - \int_M \|\mathrm{grad}\, f\|^2 e^{-\varphi},
\end{aligned}$$

which implies

$$\frac{\int_M \|\mathrm{grad}\, f\|^2 e^{-\varphi}}{\int_M f^2 e^{-\varphi}} \ge \inf_M (\mathrm{div}\, X - \|X\|^2 - c^+\|X\|).$$

Then, by taking supremum to both sides of the above inequality over $\mathscr{W}^{1,1}(M)$, we have

$$\frac{\int_M \|\mathrm{grad}\, f\|^2 e^{-\varphi}}{\int_M f^2 e^{-\varphi}} \ge \sup_{\mathscr{W}^{1,1}(M)} \left\{ \inf_M (\mathrm{div}\, X - \|X\|^2 - c^+\|X\|) \right\},$$

which implies (13). On the other hand, since $\int_M \mathrm{div}(|f|^pX) = 0$, by a direct calculation, one can obtain

$$
\begin{aligned}
0 = \int_M \operatorname{div}(|f|^p X) &= \int_M \langle \operatorname{grad}(|f|^p), X\rangle + \int_M |f|^p \operatorname{div} X \\
&\geq -\int_M p|f|^{p-1}\|\operatorname{grad} f\| \cdot \|X\| + \int_M |f|^p \operatorname{div} X \\
&\geq -\int_M p\left[\frac{(|f|^{p-1}\|X\|)^{p/(p-1)}}{\frac{p}{p-1}} + \frac{\|\operatorname{grad} f\|^p}{p}\right] + \int_M |f|^p \operatorname{div} X \\
&= \int_M [\operatorname{div} X - (p-1)\|X\|^{p/(p-1)}]|f|^p - \int_M \|\operatorname{grad} f\|^p \\
&\geq \inf_M [\operatorname{div} X - (p-1)\|X\|^{p/(p-1)}] \int_M |f|^p - \int_M \|\operatorname{grad} f\|^p,
\end{aligned}
$$

where the second inequality holds by applying Young's inequality. Therefore, we have

$$
\frac{\int_M \|\operatorname{grad} f\|^p}{\int_M |f|^p} \geq \inf_M [\operatorname{div} X - (p-1)\|X\|^{p/(p-1)}],
$$

and then, by taking supremum to both sides of the above inequality over $\mathscr{W}^{1,1}(M)$, we have

$$
\frac{\int_M \|\operatorname{grad} f\|^p}{\int_M |f|^p} \geq \sup_{\mathscr{W}^{1,1}(M)} \left\{ \inf_M [\operatorname{div} X - (p-1)\|X\|^{p/(p-1)}] \right\} \tag{15}
$$

which implies (14). This completes the proof of Lemma 3.

Remark 6. (1) Using an almost same method, we can get

$$
\lambda_{1,\varphi}^*(M) \geq \sup_{\mathscr{W}^{1,1}(M)} \left\{ \inf_{M \setminus F} (\operatorname{div} X - \|X\|^2 - c^+\|X\|) \right\}
$$

and

$$
\lambda_{1,p}(M) \geq \sup_{\mathscr{W}^{1,1}(M)} \left\{ \inf_{M \setminus F} [\operatorname{div} X - (p-1)\|X\|^{p/(p-1)}] \right\},
$$

where F has zero Riemannian volume.

(2) If M is compact, then, by taking infimum to the LHS of (15) over the space $\{f \mid f \in W_0^{1,p}(\Omega), f \neq 0\}$, one can get (14) directly. If M is noncompact, one can choose an exhaustion $\{\Omega_i\}_{i=1,2,3,\dots}$ with $\Omega_i \subset \Omega_j$, $i < j$, then as the compactness situation, one can obtain

$$
\lambda_{1,p}(\Omega_i) \geq \sup_{\mathscr{W}^{1,1}(\Omega_i)} \left\{ \inf_{\Omega_i} [\operatorname{div} X - (p-1)\|X\|^{p/(p-1)}] \right\},
$$

which, by applying domain monotonicity of the first eigenvalue of the p-Laplacian with vanishing Dirichlet data and taking limits to both sides of the above inequality as $i \to \infty$, implies (14).

For clarifying argument below better, we need to define functions $S_k(t)$ and $C_k(t)$ as follows.

$$S_k(t) = \begin{cases} \sin(\sqrt{k} \cdot t)/\sqrt{k}, & \text{if } k > 0, \\ t, & \text{if } k = 0, \\ \sinh(\sqrt{-k} \cdot t)/\sqrt{-k}, & \text{if } k < 0, \end{cases} \tag{16}$$

and

$$C_k(t) = S_k'(t).$$

We can prove the following.

THEOREM 3. *Let $\phi : M \to N \times \mathbb{R}$ be an n-dimensional $(n \geq 3)$ complete minimal isometric immersed submanifold, where the m-dimensional Riemannian manifold N has radial sectional curvature $K_{\gamma(t)}(\gamma'(t), \vec{v}) \leq k$, $\vec{v} \in T_{\gamma(t)}N$, $\|\vec{v}\| = 1$, $\vec{v} \perp \frac{\partial}{\partial t}$, along the minimizing geodesic $\gamma(t)$ issuing from a point $q \in N$. Let Ω be any connected component of $\phi^{-1}(B_N(q,r) \times \mathbb{R})$, where $r < \min\left\{\mathrm{inj}_N(q), \frac{\pi}{2\sqrt{k}}\right\}$ $(\pi/2\sqrt{k} = \infty$ if $k \leq 0)$, and $\mathrm{inj}_N(q)$ denotes the injectivity radius of N at the point q. Assume that φ is a real-valued smooth function on M with $\|\mathrm{grad}\, \varphi\| \leq c^+$, where c^+ is the supremum of the norm of the gradient of φ. Suppose in addition that*

- *if $|h(q,r)| < F^2 < \infty$, then $r \leq \left(\frac{C_k}{S_k}\right)^{-1} \cdot \frac{F^2}{(n-2)}$ or*
- *if $\lim\limits_{r \to \infty} h(q, r_0) = \infty$, then $r \leq \left(\frac{C_k}{S_k}\right)^{-1} \cdot \frac{h(q,r_0)}{(n-2)}$, where r_0 is chosen such that $(n-2)\frac{C_k(r_0)}{S_k(r_0)} - h(q, r_0) = 0$.*

Then we have

$$\lambda_{1,\varphi}^*(\Omega) \geq \left[\frac{(n-2)\frac{C_k(r)}{S_k(r)} - h(q,r) - c^+}{2}\right]^2$$

provided $c^+ < (n-2)\frac{C_k(r)}{S_k(r)} - h(q,r)$, and

$$\lambda_{1,p}(\Omega) \geq \left[\frac{(n-2)\frac{C_k(r)}{S_k(r)} - h(q,r)}{p}\right]^p.$$

PROOF. Define a function $\tilde{\rho} : N \times \mathbb{R} \to \mathbb{R}$ by $\tilde{\rho}(x,t) = \rho_N(x)$, where $\rho_N(x) = \mathrm{dist}_N(q,x)$ is the distance function in N to the point x_0. Let $\Omega \subset \phi^{-1}(B_N(q,r) \times \mathbb{R})$, $f = \tilde{\rho} \circ \phi$ and $X = \mathrm{grad}\, f$. Properly choose r such that

$\inf_\Omega \operatorname{div} X > 0$. As before, denote by Δ the Laplacian on M. Clearly, $\Delta f = \operatorname{div} X$. By Lemma 1, we have

$$\lambda^*_{1,\varphi}(\Omega) \geq \left(\frac{\inf \operatorname{div} X}{2 \sup\|X\|} - \frac{c^+}{2}\right)^2 \tag{18}$$

and

$$\lambda_{1,p}(\Omega) \geq \left(\frac{\inf \operatorname{div} X}{p \sup\|X\|}\right)^p. \tag{19}$$

Consider the orthonormal basis $\left\{\operatorname{grad}\rho_N, \frac{\partial}{\partial\theta_1}, \ldots, \frac{\partial}{\partial\theta_{m-1}}, \frac{\partial}{\partial s}\right\}$ for the tangent space $T_{(q,s)}(N \times \mathbb{R})$ with $\phi(w) = (q,s)$, where $\left\{\operatorname{grad}\rho_N, \frac{\partial}{\partial\theta_1}, \ldots, \frac{\partial}{\partial\theta_{m-1}}\right\}$ is the polar coordinates for $T_q N$. Denote by $\{e_1, e_2, \ldots, e_n\}$ an orthonormal basis for $T_w\Omega$. Then one can decompose e_i as follows

$$e_i = a_i \cdot \operatorname{grad}\rho_N + b_i \cdot \frac{\partial}{\partial s} + \sum_{j=1}^{m-1} c_i^j \cdot \frac{\partial}{\partial\theta_j}, \qquad i = 1, 2, \ldots, n,$$

where a_i, b_i, c_i^j are constants satisfying

$$a_i^2 + b_i^2 + \sum_{j=1}^{m-1} (c_i^j)^2 = 1. \tag{20}$$

By applying (12) with $Q = N \times \mathbb{R}$ to the function f, it follows that

$$\Delta f = \left[\sum_{i=1}^n \operatorname{Hess}_{N\times\mathbb{R}}\, \tilde\rho(e_i, e_i) + \langle \operatorname{grad}_{N\times\mathbb{R}}\, \tilde\rho, H\rangle\right]_{\phi(w)}, \tag{21}$$

where $H = \sum_{i=1}^n \alpha(e_i, e_i)$ is the mean curvature vector of $\phi(M)$ at the point $\phi(w)$ and the orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of $T_w M$ identified with $\{\phi_*(e_1), \phi_*(e_2), \ldots, \phi_*(e_n)\}$. By Theorem 1 and (20), we have

$$\begin{aligned}
\sum_{i=1}^n \operatorname{Hess}_{N\times\mathbb{R}}\, \tilde\rho(e_i, e_i) &= \sum_{i=1}^n \operatorname{Hess}_N\, \rho_N(e_i, e_i) \\
&= \sum_{i=1}^n \sum_{j=1}^{m-1} (c_i^j)^2 \operatorname{Hess}_N\, \rho_N\left(\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\right) \\
&\geq \sum_{i=1}^n (1 - a_i^2 - b_i^2) \frac{C_k(r)}{S_k(r)}
\end{aligned}$$

and

$$
\begin{aligned}
\langle \mathrm{grad}_{N\times\mathbb{R}}\, \tilde{\rho}, H\rangle = \langle \mathrm{grad}_N\, \rho_N, H\rangle &= \langle (\mathrm{grad}_N\, \rho_N)^{\perp}, H\rangle \\
&\le \|(\mathrm{grad}_N\, \rho_N)^{\perp}\| \cdot \|H\| = \|H\| \sqrt{1-\sum_{i=1}^{n} a_i^2} \\
&\le h(x_0, r)\sqrt{1-\sum_{i=1}^{n} a_i^2}.
\end{aligned}
$$

Substituting the above two inequalities into (21), together with the fact $1-\sum_{i=1}^{n} a_i^2 \ge 0$ and $1-\sum_{i=1}^{n} b_i^2 \ge 0$, yields

$$
\Delta f \ge (n-2)\frac{C_k(r)}{S_k(r)} - h(q,r) > 0. \tag{22}
$$

If $|h(x_0,r)| < F^2 < \infty$, then we can choose

$$
r \le \left\{ \mathrm{inj}_N(q), \frac{\pi}{2\sqrt{k}}, \left(\frac{C_k}{S_k}\right)^{-1} \cdot \frac{F^2}{(n-2)} \right\}.
$$

If $\lim_{r\to\infty} h(q,r_0) = \infty$, there exists r_0 such that $(n-2)\frac{C_k(r_0)}{S_k(r_0)} - h(q,r_0) = 0$ since $h(q,r)$ is a continuous nondecreasing function in r. Then in this situation, we can choose

$$
r \le \left\{ \mathrm{inj}_N(q), \frac{\pi}{2\sqrt{k}}, \left(\frac{C_k}{S_k}\right)^{-1} \cdot \frac{h(q,r_0)}{(n-2)} \right\}.
$$

Putting (22) with $\operatorname{div} X = \Delta f$ into (18) and (19), our estimates for $\lambda_{1,\varphi}^{*}(\Omega)$ and $\lambda_{1,p}(\Omega)$ can be obtained.

REMARK 7. If Ω is bounded and has the piecewise smooth boundary, then putting (22) with $\operatorname{div} X = \Delta f$ into (19), the estimate (14) follows. If Ω is unbounded, one can choose an exhaustion $\{\Omega_i\}_{i=1,2,3,\dots}$ with $\Omega_i \subset \Omega_j \subset \Omega$, $i<j$, and putting (22) into (19) for the bounded domain Ω_i, we have

$$
\lambda_{1,p}(\Omega_i) \ge \left[\frac{(n-2)\frac{C_k(r)}{S_k(r)} - h(q,r)}{p} \right]^p,
$$

which implies the estimate (14) by applying domain monotonicity of the first eigenvalue of the p-Laplacian with vanishing Dirichlet data and taking limits to both sides of the above inequality as $i\to\infty$. Besides, clearly, when $\varphi = const.$ or $p=2$, our estimates here are exactly the one in [2, Theorem 1.6].

5. Eigenvalue estimates for submanifolds with bounded φ-mean curvature in the hyperbolic space

For an n-dimensional ($n \geq 2$) submanifold M of the weighted manifold

$$(\mathbb{H}^m(-1), e^{-\varphi}\, dv),$$

its φ-mean curvature vector field H_φ is given by

$$H_\varphi := H + (\overline{\nabla}\varphi)^\perp$$

where $\perp$ denotes the projection onto the normal bundle of M, $\overline{\nabla}$ is the gradient operator on the hyperbolic m-space $\mathbb{H}^m(-1)$, and, as before, H is the mean curvature vector of M. We call M is *φ-minimal* if H_φ vanishes everywhere. See, e.g., [12, 19] for the notion of φ-mean curvature and some interesting applications.

Remark 8. Clearly, if $\varphi = const.$, then $H_\varphi = H$, and in this situation, "*minimal*" is equivalent to "φ-minimal". However, in general case, they are different.

Now, by applying the φ-minimal assumption and [8, Theorem 1.3], we can prove the following result.

Theorem 4. *Let M be an n-dimensional $(n \geq 2)$ complete noncompact φ-minimal submanifold of the weighted manifold $(\mathbb{H}^m(-1), e^{-\varphi}\, dv)$. If $\sup_M \|\overline{\nabla}\varphi\| < n-1$, then*

$$\lambda_{1,p}(M) \geq \left(\frac{n-1-\sup_M \|\overline{\nabla}\varphi\|}{p}\right)^p > 0. \tag{23}$$

Proof. By a direct calculation, we have

$$\sup_M \|H\| \leq \sup_M \sqrt{\|H\|^2 + \|(\overline{\nabla}\varphi)^\top\|^2} = \sup_M \|H + (\overline{\nabla}\varphi)^\top\| = \sup_M \|H_\varphi - \overline{\nabla}\varphi\|,$$

where $\top$ denotes the projection onto the tangent bundle of M. Therefore, if M is φ-minimal and $\sup_M \|\overline{\nabla}\varphi\| < n-1$, then $\sup_M \|H\| < n-1$. By applying [8, Theorem 1.3] directly, we have

$$\lambda_{1,p}(M) \geq \left(\frac{n-1-\sup_M \|H\|}{p}\right)^p > 0.$$

This implies

$$\lambda_{1,p}(M) \geq \left[\frac{n-1-\sup_M\sqrt{\|H\|^2+\|(\overline{\nabla}\varphi)^\top\|^2}}{p}\right]^p$$
$$= \left(\frac{n-1-\sup_M\|H_\varphi-\overline{\nabla}\varphi\|}{p}\right)^p$$
$$= \left(\frac{n-1-\sup_M\|\overline{\nabla}\varphi\|}{p}\right)^p > 0$$

provided M is φ-minimal and $\sup_M\|\overline{\nabla}\varphi\| < n-1$.

Remark 9. Clearly, when $\varphi = const.$, our estimate (23) becomes

$$\lambda_{1,p}(M) \geq \left(\frac{n-1}{p}\right)^p > 0,$$

which is exactly (1.5) of [8]. When $\varphi = const.$ and $p = 2$, our Theorem 4 degenerate into [6, Corollary 3].

6. Lower bounds for the first Dirichlet eigenvalues of the weighted Laplacian and the p-Laplacian on geodesic balls

For an n-dimensional $(n \geq 2)$ complete Riemannian manifold M with sectional curvature bounded from above by some constant k, Cheng [5] proved $\lambda_1(B_M(q,r)) \geq \lambda_1(B_{\mathscr{M}(n,k)}(r))$ with equality holds if and only if $B_M(q,r)$ is isometric to $B_{\mathscr{M}(n,k)}(r)$, where $B_M(q,r)$ is the geodesic ball, with center $q \in M$ and radius r, within the cut-locus of q, $B_{\mathscr{M}(n,k)}(r)$ is the geodesic ball of radius r in the n-dimensional space form $\mathscr{M}(n,k)$ with constant sectional curvature k. By using the radial sectional curvature (whose upper bound is given by a continuous function of the Riemannian distance parameter) assumption and spherically symmetric manifolds as model spaces, Freitas, Mao and Salavessa [10, Theorem 4.4] improved Cheng's conclusion mentioned above a lot. The advantage of Freitas-Mao-Salavessa's theory has been shown intuitively by numerically calculating the first Dirichlet eigenvalue of the Laplacian on torus, elliptic paraboloid and saddle (see [10, Section 6]). Besides, the principle of doing numerical calculation for the first Dirichlet eigenvalue of the Laplacian on parameterized surfaces has been given in [9, 13].

It is well-known that the first Dirichlet eigenvalue $\lambda_1(B_{\mathbb{R}^n}(r))$ of the Laplacian of a ball in $\mathbb{R}^n$ with radius r is $\lambda_1(B_{\mathbb{R}^n}(r)) = \left(\frac{J_{n/2-1}}{r}\right)^2$, where $J_{n/2-1}$ is the first zero point of the $\left(\frac{n}{2}-1\right)$-st Bessel function. By Cheng's eigenvalue comparison [5] (or its generalization [10, Theorem 4.4]), for an n-dimensional ($n \geq 2$) complete Riemannian manifold M with non-positive sectional curvature, one has

$$\lambda_1(B_M(q,r)) \geq \left(\frac{J_{n/2-1}}{r}\right)^2, \tag{24}$$

where the geodesic ball $B_M(q,r)$ is within the cut-locus of $q \in M$. The equality in (24) holds if and only if $B_M(q,r)$ is isometric to $B_{\mathbb{R}^n}(r)$.

However, applying Lemma 1, we can prove the following sharper lower bounds.

Theorem 5. *Let M be an n-dimensional ($n \geq 2$) complete manifold and a point $q \in M$. Let $B_M(q,r)$ be a geodesic ball with center $q \in M$ and radius r, where $r <$ inj(q) with inj(q) the injective radius of q. Let $\kappa(q,r) = \sup\{K_M(x) \mid x \in B_M(q,r)\}$, where $K_M(x)$ are sectional curvatures of M at x. Assume that φ is a real-valued smooth function on M with $\|\nabla\varphi\| \leq c^+$, where c^+ is the supremum of the norm of the gradient of φ. Then for $k > 0$, we have*

$$\lambda_{1,\varphi}(B_M(q,r)) \geq \begin{cases} \frac{1}{4}\cdot\left[(n-1)k\coth(kr)+\frac{1}{r}-c^+\right]^2, & \text{if } \kappa(q,r)=-k^2, \\ \left(\frac{n}{2r}-\frac{c^+}{2}\right)^2, & \text{if } \kappa(q,r)=0 \text{ and } \lambda_{1,\varphi}(M)>0, \\ \left[\frac{(n-1)kr\cot(kr)+1}{2r}-\frac{c^+}{2}\right]^2, & \text{if } \kappa(q,r)=k^2 \text{ and } r<\frac{\pi}{2k} \end{cases}$$

and

$$\lambda_{1,p}(B_M(q,r)) \geq \begin{cases} \left(\frac{1}{p}\right)^p\cdot\left[(n-1)k\coth(kr)+\frac{1}{r}\right]^p, & \text{if } \kappa(q,r)=-k^2, \\ \left(\frac{n}{pr}\right)^p, & \text{if } \kappa(q,r)=0 \text{ and } \lambda_{1,p}(M)>0, \\ \left[\frac{(n-1)kr\cot(kr)+1}{pr}\right]^p, & \text{if } \kappa(q,r)=k^2 \text{ and } r<\frac{\pi}{2k}, \end{cases}$$

where c^+ satisfies

$$c^+ < \begin{cases} (n-1)k\coth(kr)+\frac{1}{r}, & \text{if } \kappa(q,r)=-k^2, \\ \frac{n}{r}, & \text{if } \kappa(q,r)=0 \text{ and } \lambda_{1,\varphi}(M)>0, \\ \frac{(n-1)kr\cot(kr)+1}{r}, & \text{if } \kappa(q,r)=k^2 \text{ and } r<\frac{\pi}{2k}. \end{cases}$$

PROOF. As before, let ∇ and Δ be the gradient and the Laplace operators on M respectively. Choose $X = \nabla\rho^2$ with $\rho(x) = \mathrm{dist}_M(q,x)$. Then $\|X\| = 2\rho\|\nabla\rho\| = 2\rho$. By (10), we have

$$\operatorname{div} X = \Delta\rho^2 \geq 2(n-1)\rho\cdot\frac{1}{\rho}+2 = 2n, \qquad \text{if } \kappa(q,r) = 0,$$

$$\operatorname{div} X = \Delta\rho^2 \geq 2(n-1)kr\cot(kr)+2, \qquad \text{if } \kappa(q,r) = k^2,\ r<\frac{\pi}{2k},$$

and

$$\operatorname{div} X = \Delta\rho^2 \geq 2(n-1)kr\coth(kr)+2, \qquad \text{if } \kappa(q,r) = -k^2,$$

which implies

$$c(B_M(q,r)) \geq \frac{n}{r}, \qquad \text{if } \kappa(q,r) = 0,$$

$$c(B_M(q,r)) \geq \frac{(n-1)kr\cot(kr)+1}{r}, \qquad \text{if } \kappa(q,r) = k^2,\ r<\frac{\pi}{2k},$$

and

$$c(B_M(q,r)) \geq \frac{(n-1)kr\coth(kr)+1}{r}, \qquad \text{if } \kappa(q,r) = -k^2.$$

By applying Lemma 1, one can obtain estimates in Theorem 5. However, as pointed out in Remark 2, in order to use estimates in Lemma 1, one has to show $\int_{B_M(q,r)}\operatorname{div}(|f|^pX) = 0$ or $\int_{B_M(q,r)}\operatorname{div}(f^2Xe^{-\varphi}) = 0$ for all $f \in C_0^\infty(B_M(q,r))$ and the chosen vector filed X which is smooth almost everywhere in $B_M(q,r)$. This fact can be easily proven through replacing $\operatorname{div}(f^2X)$ by $\operatorname{div}(|f|^2Xe^{-\varphi})$ or $\operatorname{div}(|f|^pX)$ in the last part of the proof of [1, Theorem 4.1].

REMARK 10. If $\kappa(q,r) = -k^2$ or $\kappa(q,r) = 0$, then $\mathrm{inj}(q) = \infty$, which implies that M is noncompact. For the case of $\kappa(q,r) = -k^2$, letting $r \to \infty$, then $B_M(B(q,r))$ tends to M, and $\lambda_{1,\varphi}(M) \geq \left[\frac{(n-1)k-c^+}{2}\right]^2$ and $\lambda_{1,p}(M) \geq \left[\frac{(n-1)k}{p}\right]^p$, which are exactly the estimates given in Lemma 2. If $\varphi = const.$ (or $p = 2$) and M has non-positive sectional curvature (which satisfies assumption $\kappa(q,r) = 0$), then $\lambda_{1,\varphi}(B_M(q,r)) = \lambda_1(B_M(q,r))$ (or $\lambda_{1,p}(B_M(q,r)) = \lambda_1(B_M(q,r))$) and by Theorem 5, one has $\lambda_1(B_M(q,r)) \geq \frac{n^2}{4r^2}$, which is not so good as the estimate (24), since $J_{n/2-1} > \frac{n}{2}$ for $n \in \mathbb{N}_+$ and $n \geq 2$. However, this lower bound becomes more and more sharper as n increases, since $2J_{n/2-1}/n \to 1$ as $n \to \infty$.

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References

[1] G.-P. Bessa and J.-F. Montenegro, Eigenvalue estimates for submanifolds with locally bounded mean curvature, Ann. Glob. Anal. Geom., **24**(2) (2003), 279–290.

[2] G.-P. Bessa and J.-F. Montenegro, An extension of Barta's theorem and geometric applications, Ann. Glob. Anal. Geom., **31**(4) (2007), 345–362.

[3] G.-P. Bessa and M.-S. Costa, Eigenvalue estimates for submanifolds with locally bounded mean curvature in $N \times R$, Proc. Amer. Math. Soc., **137** (2009), 1093–1102.

[4] I. Chavel, Eigenvalues in Riemannian Geometry, Academic Press, New York, 1984.

[5] S.-Y. Cheng, Eigenfunctions and eigenvalues of the Laplacian, Am. Math. Soc. Proc. Symp. Pure Math., **27** (Part II) (1975), 185–193.

[6] L. F. Cheung and P. F. Leung, Eigenvalue estimates for submanifolds with bounded mean curvature in the hyperbolic space, Math. Z., **31** (2001), 525–530.

[7] F. Du and J. Mao, Reilly-type inequalities for the p-Laplacian on compact Riemannian manifolds, Front. Math. China, **10** (2015), 583–594.

[8] F. Du and J. Mao, Estimates for the first eigenvalue of the drifting Laplace and the p-Laplace operators on submanifolds with bounded mean curvature in the hyperbolic space, J. Math. Anal. Appl., **456** (2017), 787–795.

[9] L.-B. Hou and J. Mao, The principle of numerical calculations for eigenvalue comparison on parameterized surfaces, J. Math. Research Appl., **38** (2018), 58–62.

[10] P. Freitas, J. Mao and I. Salavessa, Spherical symmetrization and the first eigenvalue of geodesic disks on manifolds, Calc. Var. Partial Differential Equations, **51** (2014), 701–724.

[11] A. Grigor'yan, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, Bull. Amer. Math. Soc., **36**(2) (1999), 135–249.

[12] H.-Z. Li and Y. Wei, f-minimal surface and manifold with positive m-Bakry-Émery Ricci curvature, J. Geom. Anal., **25** (2015), 421–435.

[13] J. Mao, Eigenvalue estimation and some results on finite topological type, Ph.D. thesis, IST-UTL, 2013.

[14] J. Mao, Eigenvalue inequalities for the p-Laplacian on a Riemannian manifold and estimates for the heat kernel, J. Math. Pures Appl., **101**(3) (2014), 372–393.

[15] W. Meeks and H. Rosenberg, Stable minimal surfaces in $M^2 \times \mathbb{R}$, J. Differential Geom., **68**(3) (2004), 515–534.

[16] W. Meeks and H. Rosenberg, The theory of minimal surfaces in $M^2 \times \mathbb{R}$, Comment. Math. Helv., **80**(4) (2005), 811–858.

[17] H.-P. Mckean, An upper bound for the spectrum of Δ on a manifold of negative curvature, J. Differential Geom., **4** (1970), 359–366.

[18] R. Schoen and S.-T. Yau, Lectures on Differential Geometry, International Press, Boston, 1994.

[19] G. Wei and W. Wylie, Comparison geometry for the Bakry-Émery Ricci tensor, J. Differential Geom., **83** (2009), 377–405.

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