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Global attractor and Lyapunov function for one-dimensional Deneubourg chemotaxis system

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Abstract. We study the global-in-time existence and the asymptotic behavior of solutions to a one-dimensional chemotaxis system presented by Deneubourg (Insectes Sociaux 24 (1977)). The system models the self-organized nest construction process of social insects. In the limit as a time-scale coefficient tends to 0, the Deneubourg model reduces to a parabolic-parabolic Keller-Segel system with linear degradation. We first show the global-in-time existence of solutions. We next define the dynamical system of solutions and construct the global attractor. In addition, under the assumption of a large resting rate of worker insects, we construct a Lyapunov functional for the unique homogeneous equilibrium, which indicates that the global attractor consists only of the equilibrium.

1. Introduction

In the present paper, we study a chemotaxis system of three components:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - \chi \dfrac{\partial}{\partial x}\left(u \dfrac{\partial w}{\partial x}\right) + f(u) & \text{in } \Omega \times (0, \infty), \\ \delta \dfrac{\partial v}{\partial t} = -v + u & \text{in } \Omega \times (0, \infty), \\ \tau \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} - w + v & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial x}(x, t) = \dfrac{\partial w}{\partial x}(x, t) = 0 & \text{at } x = \alpha, \beta;\ t \in (0, \infty), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad w(x, 0) = w_0(x) & \text{in } \Omega. \end{cases} \tag{E}$$

Here, $\Omega = (\alpha, \beta) \subset \mathbb{R}$, $-\infty < \alpha < \beta < \infty$, is a one-dimensional bounded interval. The system (E) was presented by Deneubourg [3] (see also [2, 14])

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for modeling the self-organized nest construction process of social insects, specifically, termites. The unknown functions $u(x,t)$, $v(x,t)$, and $w(x,t)$ are the densities of, respectively, worker insects, nest building material, and a chemical substance at position x and time t. Workers deposit building material in a working area, which is expressed in the second term of the second equation of (E). It is assumed in the model (E) that a chemical substance that workers mix with the material is emitted, which is expressed in the third term of the third equation. The term $-\chi(uw_x)_x$ of the first equation represents the advection of workers due to chemotaxis, and the coefficient χ is a positive constant which indicates the intensity of chemotaxis. The function $f(u)$ consists of the migration into the working area and the resting of workers. Deneubourg [3] defined function f as

$$f(u) = 1 - \mu u \tag{1}$$

where μ is a positive constant. Here, the migration rate of workers is normalized to 1, and μ denotes the resting rate of workers. The first term of the second equation and the second term of the third equation represent the weathering of deposited materials and the decay of the chemical substance, respectively. The coefficients $\delta > 0$ and $\tau > 0$ are the time-scale constants of the reactions in the respective equations.

In the case of $\delta = 0$ and $f(u) \equiv 0$ in (E), the equations reduce to the equilibrium state $v = u$, and then the system (E) corresponds to the celebrated Keller-Segel system [7]. For a two-dimensional case, the Keller-Segel system admits blow-up solutions under suitably large product $\chi\|u_0\|_{L_1}$ of the chemotactic intensity χ and the initial total mass of biological individuals $\|u_0\|_{L_1}$ [5, 6, 12]. In addition, blow-up of solutions can occur even for sufficiently small product $\chi\|u_0\|_{L_1}$ in a higher-dimensional ball domain [21]. On the other hand, for a one-dimensional Keller-Segel system, Osaki and Yagi [15] proved the global-in-time existence of solutions and also constructed attractors, without any restriction of χ and $\|u_0\|_{L_1}$, in a one-dimensional bounded domain. Although the Deneubourg chemotaxis system (E) has three components, we expect the global-in-time existence of solutions without any restrictions on χ and $\|u_0\|_{L_1}$ in the one-dimensional case. In fact, we show in the present paper the following result.

Theorem 1. *For each triplet of nonnegative initial functions* $(u_0, v_0, w_0) \in L_2(\Omega) \times L_2(\Omega) \times H^1(\Omega)$, *the system* (E) *admits a unique global-in-time solution* (u, v, w) *in the function space*

$$\begin{cases} 0 \le u \in \mathscr{C}^1((0,\infty); H^1(\Omega)') \cap \mathscr{C}([0,\infty); L^2(\Omega)) \cap \mathscr{C}((0,\infty); H^1(\Omega)), \\ 0 \le v \in \mathscr{C}^1((0,\infty); L_2(\Omega)) \cap \mathscr{C}([0,\infty); L_2(\Omega)), \\ 0 \le w \in \mathscr{C}^1((0,\infty); L_2(\Omega)) \cap \mathscr{C}([0,\infty); H^1(\Omega)) \cap \mathscr{C}((0,\infty); H^2_N(\Omega)). \end{cases} \tag{2}$$

In addition, the solution satisfies the boundedness by the norm of initial functions such that

$$\|u(t)\|_{L_2} + \|v(t)\|_{L_2} + \|w(t)\|_{H^1} \le \psi(\|u_0\|_{L_2} + \|v_0\|_{L_2} + \|w_0\|_{H^1}), \qquad t \ge 0, \quad (3)$$

for some increasing continuous function $\psi(\cdot)$.

For the Keller-Segel system, the conservative quantity $\|u(t)\|_{L_1} = \|u_0\|_{L_1}$ has an important role. For the proof of Theorem 1, the uniform boundedness from above and below of $\|u(t)\|_{L_1}$ plays a crucial role in the same way as in the case of the Keller-Segel system (Lemma 1). There is also a modified system with a saturating effect on v, specifically, a system with the second equation of the system (E) changed to $\delta \frac{\partial v}{\partial t} = -v + \left(1 - \frac{v}{K}\right)u$. With this modification, $v_0 \le K$ implies automatically the L_∞-boundedness of $v \le K$ even in the three-dimensional case [23]. On the other hand, the system (E) does not have such a property, and therefore we should confirm the L_2-boundedness of v as a first step. This point is the essential difference from the system with a saturating effect.

Secondly, we examine the asymptotic behavior of the global solutions by defining the dynamical system. We here note that the second equation of (E) does not have any diffusion term, which generates the compactness of solution operators. This means that the dynamical system defined by Theorem 1 does not admit a compact attractor in its present form. Such a system is referred to as a *partly dissipative system* [8, 19]. We derive an inherent global attractor by decomposing the semigroup of solution operators into a compact semigroup and a perturbation vanishing as $t \to \infty$ (Theorem 4).

We finally construct a global Lyapunov functional for the constant equilibrium $(1/\mu, 1/\mu, 1/\mu)$ under a largeness condition for μ. For the two-component chemotaxis system with quadratic degradation [10] (the case of $\delta = 0$ and $f(u) = u(1 - \mu u)$ in (E), see also [16, 22, 24]), He and Zheng [4] constructed a Lyapunov functional for constant equilibrium under the condition $\mu > \chi/4$ in a two-dimensional bounded domain. On the other hand, for the first equation of (E) with a γ-th degradation $f(u) = u^{\gamma-1}(1 - \mu u)$, the same procedure as [4] derives the result that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega [\mu u - 1 - \log(\mu u)]dx \\ &\quad = -\int_\Omega \frac{u_x^2}{u^2}\,dx + \int_\Omega \chi \frac{u_x w_x}{u}\,dx - \mu^2 \int_\Omega \frac{1}{u^{2-\gamma}}(u - u^*)^2 dx. \end{aligned}$$

This shows that, although quadratic degradation $\gamma = 2$ introduces an L_2 absorbing term $-\mu^2\|u - u^*\|_{L_2}^2$, the linear degradation $\gamma = 1$ in (1) only introduces an L_1 absorbing $-\mu^2\|u\|_{L_1}$, where $u^* = 1/\mu$ is the first component of the equilibrium. To overcome this difficulty, we show the uniform boundedness of the H^1-norm of u after the passage of sufficient time. Indeed, from the embedding of $H^1(\Omega) \subset \mathscr{C}(\bar{\Omega})$, the uniform boundedness of $\|u(t)\|_{H^1} \le r$ indicates that there exists a uniform constant K_r such that $\|u(t)\|_{\mathscr{C}} \le C\|u(t)\|_{H^1} \le K_r$. We can then construct an L_2 absorbing term $-(\mu^2/K_r)\|u - u^*\|_{L_2}^2$, which shows that the Lyapunov functional is monotone decreasing for the eventual dynamical system.

The remainder of this paper is organized as follows. We first provide preliminary results that we utilize in subsequent sections. In Section 3, we show the local-in-time existence of solutions by using a semigroup method. In Section 4, we construct several a priori energy estimates by using energy methods and then give the proof of Theorem 1. In Section 5, we define the dynamical system generated by the global-in-time solutions and study the asymptotic behavior of the solutions.

Notation. Let Ω be a bounded interval in $\mathbb{R}$. For $1 \le p \le \infty$, the space of complex-valued L_p functions in Ω is denoted by $L_p(\Omega)$ with the usual norm $\|\cdot\|_{L_p}$. The complex Sobolev space in Ω of order k, $k = 0, 1, 2, \ldots$, is denoted by $H^k(\Omega)$ with norm $\|\cdot\|_{H^k}$. The Sobolev space of fractional order $s > 0$ is denoted by $H^s(\Omega)$ with norm $\|\cdot\|_{H^s}$. The space of complex-valued continuous functions on $\bar{\Omega}$ is denoted by $\mathscr{C}(\bar{\Omega})$ with norm $\|\cdot\|_{\mathscr{C}}$. Let X be a Banach space and I be an interval of $\mathbb{R}$. $\mathscr{C}(I; X)$ and $\mathscr{C}^1(I; X)$ denote the spaces of X-valued continuous functions and of X-valued continuously differentiable functions, respectively. $\mathscr{B}(I; X)$ denotes the space of X-valued bounded functions. For simplicity, we will use the universal notation C to denote various constants that are determined for each specific occurrence of Ω. In a situation where C also depends on some parameter, say η, this will be denoted by C_η. In addition, by the universal notation $\psi(\cdot)$, we will denote a continuous increasing function, which may change depending on the context.

2. Preliminaries

In this section, we shall list some well-known results in the theories of function spaces and linear operators [18, 20, 23]. Here, $\Omega = (\alpha, \beta)$ is a bounded interval in $\mathbb{R}$.

Interpolation of Sobolev spaces. For $0 \le s_0 < s < s_1 < \infty$, $H^s(\Omega)$ is the interpolation space $[H^{s_0}(\Omega), H^{s_1}(\Omega)]_\theta$ between $H^{s_0}(\Omega)$ and $H^{s_1}(\Omega)$, where $s =$

$(1-\theta)s_0 + \theta s_1$, with the estimate

$$\|w\|_{H^s} \le C\|w\|_{H^{s_0}}^{1-\theta}\|w\|_{H^{s_1}}^{\theta} \qquad \text{for } w \in H^{s_1}(\Omega). \tag{4}$$

See [23, Theorem 1.35].

Embedding theorem of Sobolev spaces. When $s > \frac{1}{2}$, $H^s(\Omega) \subset \mathscr{C}(\overline{\Omega})$ with

$$\|\cdot\|_{\mathscr{C}} \le C_s\|\cdot\|_{H^s}, \qquad s > \frac{1}{2}. \tag{5}$$

As usual, we take the identification of $L^2(\Omega)$ and its dual $L^2(\Omega)'$ and consider that $H^1(\Omega) \subset L^2(\Omega) \subset H^1(\Omega)'$. Then, (5) implies that, for any $s > \frac{1}{2}$, $L^1(\Omega) \subset H^s(\Omega)'$ with

$$\|\cdot\|_{(H^s)'} \le C_s\|\cdot\|_{L^1}, \qquad s > \frac{1}{2}. \tag{6}$$

See [23, Theorem 1.36].

Gagliardo-Nirenberg inequality. Let $1 \le q \le r \le \infty$. Then the embedding $H^1(\Omega) \cap L_q(\Omega) \subset L_r(\Omega)$ holds with the estimate

$$\|u\|_{L_r} \le C_{q,r}\|u\|_{H^1}^{a}\|u\|_{L_q}^{1-a} \qquad \text{for } u \in H^1(\Omega), \tag{7}$$

where $a = (1/q - 1/r)/(1/2 + 1/q)$.

Norms of a product of two functions. We shall use the following estimates for the product of functions. In view of (5), it holds that

$$\|uv\|_{H^m} \le C\|u\|_{H^m}\|v\|_{H^m} \qquad \text{for } u, v \in H^m(\Omega),\ m = 1, 2. \tag{8}$$

If $u \in H^1(\Omega)$ and $w \in H_N^2(\Omega)$, then

$$\langle (uw_x)_x, v\rangle_{(H^1)' \times H^1} = -(uw_x, v_x)_{L^2} \qquad \text{for } v \in H^1(\Omega).$$

Therefore, from (5),

$$\|(uw_x)_x\|_{(H^1)'} \le C\|u\|_{L^2}\|w_x\|_{L^\infty} \le C\|u\|_{L^2}\|w\|_{H^s}$$
$$\text{for } u \in H^1(\Omega),\ w \in H_N^s(\Omega),\ s > \frac{3}{2}. \tag{9}$$

Here, $H_N^s(\Omega)$ for $s > 3/2$ denotes a closed subspace of $H^s(\Omega)$ such that

$$H_N^s(\Omega) = \left\{u \in H^s(\Omega);\ \frac{du}{dx}(\alpha) = \frac{du}{dx}(\beta) = 0\right\}, \qquad s > \frac{3}{2}.$$

Domains of fractional powers of linear operator. Let $A = -(d^2/dx^2) + 1$, where d^2/dx^2 is a second-order differential operator in $L_2(\Omega)$ with the Neumann boundary condition, the domain of which is $H_N^2(\Omega)$. Then, the domains of the fractional powers of A are characterized by

$$\mathscr{D}(A^\theta) = \begin{cases} H^{2\theta}(\Omega) & \text{for } 0 \le \theta < \frac{3}{4}, \\ H_N^{2\theta}(\Omega) & \text{for } \frac{3}{4} < \theta < \frac{7}{4}, \end{cases} \tag{10}$$

with norm equivalence.

3. Local solution

We first review the existence theorem for local solutions to an abstract equation in a Banach space [23, Chap. 4] (also [17]). Let X be a Banach space with norm $\|\cdot\|_X$. We consider the following Cauchy problem for a semilinear abstract evolution equation in X:

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), \quad t > 0, \\ U(0) = U_0. \end{cases} \tag{11}$$

Here, A is a sectorial operator of X satisfying that its spectral set is contained in a sectorial domain $\Sigma = \{\lambda \in \mathbb{C}; |\arg \lambda| \le \phi\}$ for some $0 \le \phi < \pi/2$, and $\|(\lambda - A)^{-1}\|_{\mathscr{L}(X)} \le M/(|\lambda| + 1)$, where $\lambda \notin \Sigma$ and M is a constant. The nonlinear operator F is a mapping from $\mathscr{D}(A^\eta)$ to X, where $0 < \eta < 1$, and it also satisfies a Lipschitz condition:

$$\begin{aligned} \|F(U) - F(\tilde{U})\|_X \le \psi(\|A^\gamma U\|_X + \|A^\gamma \tilde{U}\|_X) \times [&\|A^\eta(U - \tilde{U})\|_X \\ &+ (\|A^\eta U\|_X + \|A^\eta \tilde{U}\|_X)\|A^\gamma(U - \tilde{U})\|_X], \\ &U, \tilde{U} \in \mathscr{D}(A^\eta), \end{aligned} \tag{12}$$

where γ is an exponent such that $0 < \gamma \le \eta < 1$, and $\psi(\cdot)$ is some increasing continuous function. The initial value U_0 is taken in $\mathscr{D}(A^\gamma)$. Then, from [23, Theorem 4.1] (or [17, Theorem 3.1]), we have the existence theorem of the local solutions to (11):

Proposition 1 ([23, Theorem 4.1]). *Under the above assumptions, for any $U_0 \in \mathscr{D}(A^\gamma)$, the equation* (11) *possesses a unique local solution in the function space:*

$$\begin{cases} U \in \mathscr{C}^1((0, T_{U_0}]; X) \cap \mathscr{C}([0, T_{U_0}]; \mathscr{D}(A^\gamma)) \cap \mathscr{C}((0, T_{U_0}]; \mathscr{D}(A)), \\ t^{1-\gamma} U \in \mathscr{B}((0, T_{U_0}]; \mathscr{D}(A)) \end{cases}$$

with the estimate

$$t^{1-\gamma}\|AU(t)\|_X + \|A^\gamma U(t)\|_X \le C_{U_0}, \qquad 0 < t \le T_{U_0}.$$

Here, T_{U_0} and C_{U_0} are positive constants depending only on the norm $\|A^\gamma U_0\|_X$. In addition, the mapping $U_0 \mapsto U(t)$ is continuous in $\mathscr{D}(A^\gamma)$.

By applying Proposition 1, we can show the existence of the local-in-time solutions to (E).

Theorem 2. *For each triplet of initial functions $(u_0, v_0, w_0) \in L_2(\Omega) \times L_2(\Omega) \times H^1(\Omega)$, the problem* (E) *admits a unique local-in-time solution (u, v, w) in the function space*

$$\begin{cases} u \in \mathscr{C}^1((0,T]; H^1(\Omega)') \cap \mathscr{C}([0,T]; L_2(\Omega)) \cap \mathscr{C}((0,T]; H^1(\Omega)), \\ v \in \mathscr{C}^1((0,T]; L_2(\Omega)) \cap \mathscr{C}([0,T]; L_2(\Omega)), \\ w \in \mathscr{C}^1((0,T]; L_2(\Omega)) \cap \mathscr{C}([0,T]; H^1(\Omega)) \cap \mathscr{C}((0,T]; H_N^2(\Omega)) \end{cases} \tag{13}$$

with the estimate

$$\begin{aligned} &t^{1/2}\{\|u(t)\|_{H^1} + \|w(t)\|_{H^2}\} + \{\|u(t)\|_{L_2} + \|v(t)\|_{L_2} + \|w(t)\|_{H^1}\} \\ &\quad \le C, \qquad 0 < t \le T, \end{aligned} \tag{14}$$

where T and C are positive constants depending only on the norm $\|u_0\|_{L_2} + \|v_0\|_{L_2} + \|w_0\|_{H^1}$. In addition, the mapping $(u_0, v_0, w_0) \mapsto (u(t), v(t), w(t))$ is continuous in $L_2(\Omega) \times L_2(\Omega) \times H^1(\Omega)$.

Proof. The system (E) can be expressed as a semilinear parabolic equation

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), & t > 0, \\ U(0) = U_0 = {}^T[u_0 \;\; v_0 \;\; w_0] \end{cases} \tag{15}$$

in a product Banach space

$$X = H^1(\Omega)' \times L_2(\Omega) \times L_2(\Omega).$$

Here, we define the linear operator A by

$$A = \begin{bmatrix} -\frac{d^2}{dx^2} + 1 & 0 & 0 \\ 0 & \delta^{-1} & 0 \\ 0 & 0 & \tau^{-1}\left(-\frac{d^2}{dx^2} + 1\right) \end{bmatrix}, \qquad \mathscr{D}(A) = H^1(\Omega) \times L_2(\Omega) \times H_N^2(\Omega).$$

The nonlinear operator F is defined by

$$F(U) = \begin{bmatrix} -\chi(uw_x)_x + f(u) + u \\ \delta^{-1}u \\ \tau^{-1}v \end{bmatrix},$$

$$U = {}^T[u \ v \ w] \in \mathscr{D}(A^\eta) = H^{3/4}(\Omega) \times L_2(\Omega) \times H_N^{7/4}(\Omega)$$

with $\eta = \frac{7}{8}$. Here, $f(u) = 1 - \mu u$ for $u \in \mathbf{C}$. The initial value U_0 is taken in the function space

$$\mathscr{D}(A^\gamma) = L_2(\Omega) \times L_2(\Omega) \times H^1(\Omega)$$

with $\gamma = \frac{1}{2}$. Under this setting, we need to verify only the Lipschitz condition (12). Let $U = {}^T[u \ v \ w]$, $\tilde{U} = {}^T[\tilde{u} \ \tilde{v} \ \tilde{w}] \in \mathscr{D}(A^\eta)$. Then, we have

$$\begin{aligned} \|F(U) - F(\tilde{U})\|_X \le{} & \chi\|(uw_x - \tilde{u}\tilde{w}_x)_x\|_{(H^1)'} + (\mu+1)\|u - \tilde{u}\|_{(H^1)'} \\ & + \delta^{-1}\|u - \tilde{u}\|_{L_2} + \tau^{-1}\|v - \tilde{v}\|_{L_2}. \end{aligned}$$

For the first term of the right-hand side, we have from (6) with $s = 3/4$

$$\begin{aligned} \|(uw_x - \tilde{u}\tilde{w}_x)_x\|_{(H^1)'} &\le C(\|u - \tilde{u}\|_{L_2}\|w_x\|_{H^{3/4}} + \|\tilde{u}\|_{L_2}\|(w - \tilde{w})_x\|_{H^{3/4}}) \\ &\le C(\|\tilde{u}\|_{L_2}\|w - \tilde{w}\|_{H^{7/4}} + \|w\|_{H^{7/4}}\|u - \tilde{u}\|_{L_2}). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \|F(U) - F(\tilde{U})\|_X &\le \chi C(\|\tilde{u}\|_{L_2}\|w - \tilde{w}\|_{H^{7/4}} + \|w\|_{H^{7/4}}\|u - \tilde{u}\|_{L_2}) \\ &\quad + (\mu+1)\|u - \tilde{u}\|_{L_2} + \delta^{-1}\|u - \tilde{u}\|_{L_2} + \tau^{-1}\|v - \tilde{v}\|_{L_2} \\ &\le [1 + \mu + \delta^{-1} + \tau^{-1} + \chi C(1 + \|\tilde{u}\|_{L_2})] \\ &\quad \times [\|w - \tilde{w}\|_{H^{7/4}} + (\|w\|_{H^{7/4}} + 1)(\|u - \tilde{u}\|_{L_2} + \|v - \tilde{v}\|_{L_2})]. \end{aligned}$$

Thus, we have verified (12); we have completed the proof. □

Proposition 2. *Under the assumptions in Theorem* 2, *if* $u_0 \ge 0$, $v_0 \ge 0$, *and* $w_0 \ge 0$, *then the solution* (u, v, w) *also satisfies* $u(t) \ge 0$, $v(t) \ge 0$, *and* $w(t) \ge 0$ *for* $0 \le t \le T$.

Proof. We first note that the solution (u, v, w) is real valued. Indeed, the complex conjugate $(\bar{u}, \bar{v}, \bar{w})$ is also a solution to (E). From the uniqueness of the solution, we have $(u, v, w) = (\bar{u}, \bar{v}, \bar{w})$. We shall show the nonnegativity of the local solutions. Let $H(u)$ be a decreasing $\mathscr{C}^3$ function defined for $u \in (-\infty, \infty)$ such that $H(u) > 0$ for $u < 0$ and $H(u) = 0$ for $u \ge 0$. Moreover, let

$H(u)$ satisfy the following conditions:

$$0 \le H'(u)u \le CH(u), \qquad u \in (-\infty, \infty),$$
$$0 \le H''(u)u^2 \le CH(u), \qquad u \in (-\infty, \infty)$$

with some constant C (For example, $H(u) = \frac{1}{4}u^4$ for $u < 0$; 0 for $u \ge 0$). Then, the function

$$\varphi(t) = \int_\Omega H(u(x,t))dx, \qquad 0 \le t \le T,$$

is a nonnegative $\mathscr{C}^1$-function with the derivative

$$\begin{aligned}\varphi'(t) &= \int_\Omega H'(u)[u_{xx} - \chi(uw_x)_x + f(u)]dx \\ &= -\int_\Omega H''(u)u_x^2\,dx + \chi\int_\Omega H''(u)u_x \cdot uw_x\,dx + \int_\Omega H'(u)f(u)dx.\end{aligned}$$

Here we note that $\int_\Omega H'(u)f(u)dx \le 0$. The integral term from the chemotaxis term can be estimated as

$$\chi\int_\Omega uH''(u)u_x \cdot w_x\,dx \le \frac{1}{2}\int_\Omega H''(u)u_x^2\,dx + \frac{\chi^2}{2}\int_\Omega u^2w_x^2H''(u)dx.$$

From Theorem 2, we have for the second term

$$\begin{aligned}\frac{\chi^2}{2}\int_\Omega u^2w_x^2H''(u)dx &\le C\|w_x\|_{H^{3/4}}^2\int_\Omega u^2H''(u)dx \le C\|w\|_{H^{7/4}}^2\int_\Omega H(u)dx \\ &\le C\|w\|_{H^1}^{1/2}\|w\|_{H^2}^{3/2}\int_\Omega H(u)dx \le Ct^{-3/4}\int_\Omega H(u)dx.\end{aligned}$$

This indicates that $\varphi'(t) \le Ct^{-3/4}\varphi(t)$, and consequently $\varphi(t) \le \exp(Ct^{1/4})\varphi(0)$. Then, $u_0(x) \ge 0$ gives that $\varphi(t) = 0$, that is, $u(x,t) \ge 0$ for $0 \le t \le T$. The nonnegativity of $v(x,t)$ and $w(x,t)$ can be also proved by the comparison principle. □

4. A priori estimates and global-in-time solutions

Proposition 3. *Let* (u, v, w) *be a local-in-time solution to* (E). *Then, it holds that*

$$\|u(t)\|_{L_1} = \int_\Omega u(x,t)dx = e^{-\mu t}\left(\|u_0\|_{L_1} - \frac{|\Omega|}{\mu}\right) + \frac{|\Omega|}{\mu}, \qquad t \ge 0. \tag{16}$$

In addition,

$$0 \le \min\left\{\|u_0\|_{L_1}, \frac{|\Omega|}{\mu}\right\} < \|u(t)\|_{L_1} \le \max\left\{\|u_0\|_{L_1}, \frac{|\Omega|}{\mu}\right\} := K_0, \qquad t > 0.$$

Proof. Integrating the first equation of (E) over Ω, we have

$$\frac{d}{dt}\int_\Omega u\,dx = \int_\Omega f(u)dx = |\Omega| - \mu\int_\Omega u\,dx. \tag{17}$$

By solving this in $\|u(t)\|_{L_1}$, we obtain (16). Also, this provides the lower and upper estimates for $\|u(t)\|_{L_1}$. □

Proposition 4. *Let (u, v, w) be a local-in-time solution to* (E). *Then, it holds that*

$$\|u(t)\|_{L_2} + \|v(t)\|_{L_2} + \|w(t)\|_{H^1} \le \psi(\|u_0\|_{L_2} + \|v_0\|_{L_2} + \|w_0\|_{H^1}), \qquad 0 \le t \le T, \tag{18}$$

where $\psi(\cdot)$ is some increasing continuous function.

Proof. Multiplying the first equation of (E) by u and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx + \int_\Omega u_x^2\,dx + \mu\int_\Omega u^2\,dx \le \frac{\chi}{2}\int_\Omega u^2|w_{xx}|dx + \|u\|_{L_1}.$$

Here, we put

$$q(K_0) = \frac{\nu_0}{C_{1,4}^4\chi^2K_0^2}, \qquad \nu_0 = \min\{1, \mu\},$$

where K_0 is the supremum of $\|u(t)\|_{L_1}$, and $C_{1,4}$ is an embedding constant of the Gagliardo-Nirenberg inequality (7) with $r = 4$ and $q = 1$. Then, the integral from the chemotaxis term can be estimated as

$$\chi\int_\Omega u^2|w_{xx}|dx \le \chi\|u\|_{L_4}^2\|w_{xx}\|_{L_2} \le \chi C_{1,4}^2\|u\|_{H^1}\|u\|_{L_1}\|w_{xx}\|_{L_2}$$
$$\le \nu_0\|u\|_{H^1}^2 + \frac{1}{4q}\|w_{xx}\|_{L_2}^2.$$

Then, we have

$$\frac{d}{dt}\int_\Omega u^2\,dx + \int_\Omega u_x^2\,dx + \mu\int_\Omega u^2\,dx \le \frac{1}{4q}\|w_{xx}\|_{L_2}^2 + 2\|u\|_{L_1}. \tag{19}$$

Meanwhile, multiplying the second equation of (E) by v and integrating over Ω, we have

$$\frac{\delta}{2}\frac{d}{dt}\int_\Omega v^2\,dx + \frac{1}{2}\int_\Omega v^2\,dx \le \frac{1}{2}\int_\Omega u^2\,dx \le \frac{C_{1,2}^2}{2}\|u\|_{L_1}^{4/3}\|u\|_{H^1}^{2/3}$$
$$\le \frac{\nu_0 q}{8}\|u\|_{H^1}^2 + C\frac{\chi}{\nu_0}K_0^3. \tag{20}$$

Similarly, multiplying the third equation of (E) by $(-w_{xx}+w)$ and integrating over Ω, we have

$$\frac{\tau}{2}\frac{d}{dt}\int_\Omega (w^2+w_x^2)dx + \frac{1}{2}\int_\Omega (w_{xx}^2+w_x^2+w^2)dx \le \int_\Omega v^2\,dx. \tag{21}$$

Multiplying (19) by q, (20) by 2, and (21) by 1/2, and adding all of these up, we obtain

$$\frac{d}{dt}\left[q(K_0)\|u(t)\|_{L_2}^2 + \delta\|v(t)\|_{L_2}^2 + \frac{\tau}{4}\|w(t)\|_{H^1}^2\right]$$
$$+\eta_0\left[q(K_0)\|u\|_{L_2}^2 + \delta\|v\|_{L_2}^2 + \frac{\tau}{4}\|w\|_{H^1}^2\right]$$
$$\le 2q(K_0)\|u\|_{L_1} + C\frac{\chi}{\nu_0}K_0^3 \le C\left(\frac{\nu_0}{\chi^2 K_0} + \frac{\chi}{\nu_0}K_0^3\right),$$

where

$$\eta_0 = \min\left\{\frac{\nu_0}{2}, \frac{1}{2\delta}, \frac{1}{\tau}\right\} = \min\left\{\frac{1}{2}, \frac{\mu}{2}, \frac{1}{2\delta}, \frac{1}{\tau}\right\}.$$

Solving this, we obtain that

$$q(K_0)\|u(t)\|_{L_2}^2 + \delta\|v(t)\|_{L_2}^2 + \frac{\tau}{4}\|w(t)\|_{H^1}^2$$
$$\le e^{-\eta_0 t}\left[q(K_0)\|u_0\|_{L_2}^2 + \delta\|v_0\|_{L_2}^2 + \frac{\tau}{4}\|w_0\|_{H^1}^2\right] + \frac{C}{\eta_0}\left(\frac{\nu_0}{\chi^2 K_0} + \frac{\chi}{\nu_0}K_0^3\right). \tag{22}$$

This shows that $\|u(t)\|_{L_2}$ is estimated from above by $\psi(\|u_0\|_{L_2} + \|v_0\|_{L_2} + \|w_0\|_{H^1})$. The norms of $\|v(t)\|_{L_2}$ and $\|w(t)\|_{L_2}$ can be also estimated from above by $\psi(\|u_0\|_{L_2} + \|v_0\|_{L_2} + \|w_0\|_{H^1})$ by solving again (20) and (21). □

Proposition 4 indicates the global-in-time existence of solutions to (E):

Proof of Theorem 1. From Theorem 2 and Proposition 2, for each triplet of nonnegative initial functions (u_0, v_0, w_0), there exists a unique nonnegative local solution (u, v, w) on an interval $[0, T]$, where the existence time T depends only on the norm of initial function $\|u_0\|_{L_2} + \|v_0\|_{L_2} + \|w_0\|_{H^1}$. In addition, from Proposition 4, the norm $\|u(t)\|_{L_2} + \|v(t)\|_{L_2} + \|w(t)\|_{H^1}$, $0 \le t \le T$, is estimated from above by a uniform constant $\psi(\|u_0\|_{L_2} + \|v_0\|_{L_2} + \|w_0\|_{H^1})$ which depends only on the norm $\|u_0\|_{L_2} + \|v_0\|_{L_2} + \|w_0\|_{H^1}$. Hence, the existence interval can be extended to $[0, T + \tilde{T}]$, and the norm $\|u(t)\|_{L_2} + \|v(t)\|_{L_2} + \|w(t)\|_{H^1}$, $0 \le t \le T + \tilde{T}$, is estimated again by the same constant $\psi(\|u_0\|_{L_2} + \|v_0\|_{L_2} + \|w_0\|_{H^1})$, from Proposition 4. Then, the existence time can be extended to $T + 2\tilde{T}$. Iteration of this procedure proves the global-in-time existence of solutions with the boundedness (3). □

In the last part of this section, we shall construct higher order a priori estimates for the local-in-time solutions by taking higher order initial functions.

Proposition 5. *Let (u, v, w) be a local-in-time solution to* (E). *Then, for an arbitrary number $t_0 \in (0, T]$, it holds that*

$$\|w_{xx}(t)\|_{L_2}^2 \le 2\tau^2 e^{-(1/\tau)(t-t_0)}\|w_t(t_0)\|_{L_2}^2 + \frac{4\tau}{\delta^2}\int_{t_0}^t e^{-(1/\tau)(t-s)}(\|u(s)\|_{L_2}^2 + \|v(s)\|_{L_2}^2)ds + 4\|v(t)\|_{L_2}^2, \qquad t_0 \le t \le T. \tag{23}$$

Proof. Applying operator $\partial/\partial t$ to the third equation of (E) and multiplying by w_t, and then integrating over Ω, we have

$$\frac{\tau}{2}\frac{d}{dt}\|w_t\|_{L_2}^2 + \|w_{xt}\|_{L_2}^2 + \frac{1}{2}\|w_t\|_{L_2}^2 \le \frac{1}{2}\|v_t\|_{L_2}^2 \le \frac{1}{\delta^2}(\|u\|_{L_2}^2 + \|v\|_{L_2}^2).$$

Solving this in $\|w_t(t)\|_{L_2}^2$ from t_0 to t, we have

$$\|w_t\|_{L_2}^2 \le e^{-(1/\tau)(t-t_0)}\|w_t(t_0)\|_{L_2}^2 + \frac{2}{\tau\delta^2}\int_{t_0}^t e^{-(1/\tau)(t-s)}(\|u(s)\|_{L_2}^2 + \|v(s)\|_{L_2}^2)ds, \qquad t_0 \le t \le T.$$

Since

$$\begin{aligned}\|w_{xx}\|_{L_2}^2 &= \tau^2\|w_t\|_{L_2}^2 - \|w\|_{L_2}^2 - \|v\|_{L_2}^2 \\ &\quad + 2\int_\Omega w w_{xx}\,dx - 2\int_\Omega v w_{xx}\,dx + 2\int_\Omega v w\,dx \\ &\le \tau^2\|w_t\|_{L_2}^2 + \frac{1}{2}\|w_{xx}\|_{L_2}^2 + 2\|v\|_{L_2}^2,\end{aligned}$$

we obtain the estimate (23). □

Proposition 6. *Let* (u, v, w) *be a local-in-time solution to* (E), *and* $t_0 \in (0, T]$ *be arbitrarily fixed. Then,*

$$\|u_x\|_{L_2}^2 + \delta\|v\|_{L_2}^2 \le e^{-\tilde{\eta}_0(t-t_0)}(\|u(t_0)\|_{H^1}^2 + \delta\|v(t_0)\|_{L_2}^2) + \int_{t_0}^{t} e^{-\tilde{\eta}_0(t-s)}\|u(s)\|_{L_2}^2$$
$$\times\left(1 + v_0 + \frac{C\chi^4}{v_0}\|w(s)\|_{H^2}^4\right)ds, \qquad t_0 \le t \le T,$$

where $v_0 = \min\{1, \mu\}$, $\tilde{\eta}_0 = \min\{\mu, \frac{1}{\delta}\}$.

Proof. Multiplying the first equation of (E) by u_{xx}, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_x^2\,dx + \mu\int_\Omega u_x^2\,dx + \int_\Omega u_{xx}^2\,dx = \chi\int_\Omega u_{xx}(u_x w_x + u w_{xx})dx$$
$$\le \frac{1}{2}\int_\Omega u_{xx}^2\,dx + \chi^2\int_\Omega (u_x^2 w_x^2 + u^2 w_{xx}^2)dx.$$

The two terms coming from the chemotaxis term can be estimated as follows:

$$\chi^2\int_\Omega u_x^2 w_x^2\,dx \le \chi^2\|w_x\|_{L_\infty}^2\|u_x\|_{L_2}^2 \le C\chi^2\|w\|_{H^2}^2\|u\|_{H^2}\|u\|_{L_2}$$
$$\le \frac{v_0}{4}\|u\|_{H^2}^2 + \frac{C\chi^4}{v_0}\|u\|_{L_2}^2\|w\|_{H^2}^4,$$
$$\chi^2\int_\Omega u^2 w_{xx}^2\,dx \le \chi^2\|u\|_{L_\infty}^2\|w_{xx}\|_{L_2}^2 \le C\chi^2\|u\|_{H^1}\|u\|_{L_2}\|w\|_{H^2}^2$$
$$\le \frac{v_0}{4}\|u\|_{H^1}^2 + \frac{C\chi^4}{v_0}\|u\|_{L_2}^2\|w\|_{H^2}^4$$

with $v_0 = \min\{1, \mu\}$. We then obtain

$$\frac{d}{dt}\|u_x\|_{L_2}^2 + \mu\|u_x\|_{L_2}^2 \le \|u\|_{L_2}^2\left(v_0 + \frac{C\chi^4}{v_0}\|w\|_{H^2}^4\right). \tag{24}$$

Next, multiplying the second equation of (E) by v, and integrating over Ω, we have

$$\frac{\delta}{2}\frac{d}{dt}\|v\|_{L_2}^2 + \frac{1}{2}\|v\|_{L_2}^2 \le \frac{1}{2}\|u\|_{L_2}^2. \tag{25}$$

Multiplying (25) by 2 and adding the result to (24), we have

$$\frac{d}{dt}[\|u_x\|_{L_2}^2 + \delta\|v\|_{L_2}^2] + \tilde{\eta}_0[\|u_x\|_{L_2}^2 + \delta\|v\|_{L_2}^2]$$
$$\leq \|u\|_{L_2}^2\left(1 + v_0 + \frac{C\chi^4}{v_0}\|w\|_{H^2}^4\right).$$

where $\tilde{\eta}_0 = \min\{\mu, \frac{1}{\delta}\}$. Solving this from t_0 to t, we obtain the estimate. □

5. Dynamical systems, attractors, and a Lyapunov function

We study the asymptotic behavior of the global solutions obtained in Theorem 1. Let

$$\mathscr{H} = H^1(\Omega)' \times L_2(\Omega) \times L_2(\Omega)$$

be the universal space of a dynamical system. We set the initial function space as

$$\mathscr{K} = \{{}^T[u_0 \ \ v_0 \ \ w_0] \in L_2(\Omega) \times L_2(\Omega) \times H^1(\Omega);\ u_0, v_0, w_0 > 0\},$$
$$\|U_0\|_{\mathscr{K}} = \|u_0\|_{L_2} + \|v_0\|_{L_2} + \|w_0\|_{H^1}, \qquad U_0 = {}^T[u_0 \ \ v_0 \ \ w_0].$$

Theorem 1 with the strong comparison principle (e.g. [9, p. 331]) defines a continuous semigroup of the solution operator $S(t) : \mathscr{K} \to \mathscr{K}$. We will consider the dynamical system $(S(t), \mathscr{K}, \mathscr{H})$ hereafter.

THEOREM 3. *A ball $\mathscr{B}$ in $\mathscr{K}$ with sufficiently large radius r,*

$$\mathscr{B} := \{(u, v, w) \in H^1(\Omega) \times L_2(\Omega) \times H_N^2(\Omega);$$
$$\|u\|_{H^1} + \|v\|_{L_2} + \|w\|_{H^2} \leq r,\ u, v, w > 0\} \subset \mathscr{K},$$

is an absorbing set for the dynamical system $(S(t), \mathscr{K}, \mathscr{H})$. In addition, the radius r of the ball $\mathscr{B}$ is of order $O(1)$ for large μ.

PROOF. Propositions 5 and 6 directly show the theorem, using the estimate (26) in Lemma 2 below. □

To complete the proof of Theorem 3, we provide two lemmas.

LEMMA 1. *Let B be an arbitrary bounded set of $\mathscr{K}$. Then, for any $U_0 \in B$, there exists a uniform time t_B in B such that*

$$\frac{|\Omega|}{2\mu} \leq \|u(t)\|_{L_1} \leq \frac{2|\Omega|}{\mu} \qquad \text{for all } t \geq t_B.$$

PROOF. From Proposition 3, if $\|u_0\|_{L_1} > \frac{|\Omega|}{\mu}$, then for all $t \geq t_1 := \mu^{-1} \log(\Omega^{-1}\mu\|u_0\|_{L_1} - 1)$, it holds that $\|u(t)\|_{L_1} \leq \frac{2|\Omega|}{\mu}$. Since

$$\|u_0\|_{L_1} \leq |\Omega|^{1/2}\|u_0\|_{L_2} \leq C_B,$$

we can choose

$$t_B = \mu^{-1} \log(\Omega^{-1}\mu C_B - 1) \geq t_1,$$

which shows the conclusion. The other case is proved in a similar manner. □

LEMMA 2. *Let B be an arbitrary bounded set of $\mathscr{K}$. Then, for any $U_0 \in B$, there exist a uniform time t_B in B and a uniform constant r_0, independent of B, such that*

$$\|U(t)\|_{\mathscr{K}} \leq r_0, \qquad t \geq t_B. \tag{26}$$

In addition, $r_0 = O(1)$ for large μ.

PROOF. Let t_B be a time obtained in Lemma 1. Then, set

$$q(\|u(t)\|_{L_1}) = \frac{\nu_0}{C_{1,4}^4\chi^2\|u(t)\|_{L_1}^2}, \qquad t > 0.$$

Consider again the inequalities (19) with $q = q(\|u(t)\|_{L_1})$ and (21), as well as, in place of (20),

$$\frac{\delta}{2}\frac{d}{dt}\int_\Omega v^2\,dx + \frac{1}{2}\int_\Omega v^2\,dx \leq \frac{\nu_0 q}{8}\|u\|_{H^1}^2 + C\frac{\chi}{\nu_0}\|u\|_{L_1}^3. \tag{27}$$

Since it is clear that

$$\begin{aligned}\frac{dq}{dt}\|u\|_{L_2}^2 &= -\frac{2\nu_0}{C_{1,4}^4\chi^2\|u\|_{L_1}^3}(|\Omega| - \mu\|u(t)\|_{L_1})\|u\|_{L_2}^2 \\ &\leq 2C_{1,2}^2\mu q\|u\|_{L_1}^{4/3}\|u\|_{H^1}^{2/3} \leq \frac{\nu_0 q}{4}\|u\|_{H^1}^2 + C\frac{\sqrt{\nu_0}\mu^{3/2}}{\chi^2},\end{aligned} \tag{28}$$

multiplying (19) by q, (27) by 2, and (21) by $1/2$, and adding all of these to (28), by noting that $\frac{d}{dt}(q\|u\|_{L_2}^2) = q\frac{d}{dt}\|u\|_{L_2}^2 + \frac{dq}{dt}\|u\|_{L_2}^2$, we obtain that

$$\begin{aligned}&\frac{d}{dt}\left[q(\|u(t)\|_{L_1})\|u(t)\|_{L_2}^2 + \delta\|v(t)\|_{L_2}^2 + \frac{\tau}{4}\|w(t)\|_{H^1}^2\right] \\ &\quad + \eta_0\left[q(\|u(t)\|_{L_1})\|u\|_{L_2}^2 + \delta\|v\|_{L_2}^2 + \frac{\tau}{4}\|w\|_{H^1}^2\right] \\ &\leq 2q(\|u(t)\|_{L_1})\|u\|_{L_1} + C\frac{\chi}{\nu_0}\|u\|_{L_1}^3 + C\frac{\sqrt{\nu_0}}{\chi^2}\mu^{3/2}.\end{aligned}$$

Solving this from t_B to t, we obtain that

$$
\begin{aligned}
& q(\|u(t)\|_{L_1})\|u(t)\|_{L_2}^2 + \delta\|v(t)\|_{L_2}^2 + \frac{\tau}{4}\|w(t)\|_{H^1}^2 \\
& \quad \le e^{-\eta_0(t-t_B)}\left[q(\|u(t_B)\|_{L_1})\|u(t_B)\|_{L_2}^2 + \delta\|v(t_B)\|_{L_2}^2 + \frac{\tau}{4}\|w(t_B)\|_{H^1}^2\right] \\
& \qquad + C\int_{t_B}^{t} e^{-\eta_0(t-s)}\left(\frac{v_0}{\chi^2\|u(s)\|_{L_1}} + \frac{\chi}{v_0}\|u(s)\|_{L_1}^3 + \frac{\sqrt{v_0}\mu^{3/2}}{\chi^2}\right)ds. \qquad (29)
\end{aligned}
$$

By taking t_B again sufficiently large, we obtain from the uniform estimate of $\|u(t)\|_{L_1}$ in Lemma 1 that

$$
\frac{v_0\mu^2}{4C_{1,4}^4\chi^2|\Omega|^2}\|u(t)\|_{L_2}^2 \le 1 + \frac{C}{\eta_0}\left(\frac{v_0\mu}{\chi^2} + \frac{\chi}{v_0\mu^3} + \frac{\sqrt{v_0}\mu^{3/2}}{\chi^2}\right), \qquad t \ge t_B.
$$

This shows that $\|u(t)\|_{L_2}$ is bounded above independently of B for $t \ge t_B$. In addition, the upper bound of $\|u(t)\|_{L_2}$ is of order $O(\mu^{-1/4})$ for large μ. By using this, we next show the uniform estimate of $\|v(t)\|_{L_2}$, instead of (29). Indeed, from the differential inequality (20), there exists another large time t_B, depending only on B, such that

$$
\|v(t)\|_{L_2}^2 \le e^{-t/\delta}\|v_0\|_{L_2}^2 + \int_0^t e^{-(1/\delta)(t-s)}\|u(s)\|_{L_2}^2 ds \le 1 + \psi(\mu^{-1/2}), \qquad t \ge t_B.
$$

Similarly, from the differential inequality (21), by taking t_B once again sufficiently large, we have

$$
\|w(t)\|_{H^1}^2 \le e^{-t/\tau}\|w_0\|_{H^1}^2 + \int_0^t e^{-(1/\tau)(t-s)}\|v(s)\|_{L_2}^2 ds \le r_0, \qquad t \ge t_B,
$$

where $r_0 = O(1)$ for large μ. □

We decompose the second component as $v(t) = v_1(t) + v_2(t)$ where

$$
v_1(t) = \int_0^t e^{-(1/\delta)(t-s)}u(s)ds \qquad \text{and} \qquad v_2(t) = e^{-(1/\delta)t}v_0.
$$

We then also decompose the solution operator $S(t)$ into a compact operator $S_1(t)$ and the perturbation $S_2(t)$ such that $S(t) = S_1(t) + S_2(t)$, where $S_1(t) : (u_0, v_0, w_0) \mapsto (u(t), v_1(t), w(t))$ and $S_2(t) : (u_0, v_0, w_0) \mapsto (0, v_2(t), 0)$. From $\|v_2(t)\|_{L_2} = e^{-(1/\delta)t}\|v_0\|_{L_2}$, we have that for every bounded set $B \subset \mathscr{K}$,

$$
\sup_{U_0 \in B}\|S_2(t)U_0\|_{\mathscr{K}} \to 0, \qquad t \to \infty.
$$

Meanwhile, we can show that all orbits of $v_1(t)$ with $U_0 \in B$ are uniformly compact in $\mathscr{K}$. Specifically, we have from Theorem 2 and Propositions 4, 5, and 6 that

$$\|v_1(t)\|_{H^1} \le \int_0^t e^{-(1/\delta)(t-s)} \|u(s)\|_{H^1} ds$$
$$\le \psi(\|U_0\|_{\mathscr{K}}) \int_0^t e^{-(1/\delta)(t-s)} (s^{-1/2} + 1) ds \le \psi(\|U_0\|_{\mathscr{K}}).$$

This implies that $\bigcup_{t \ge t_B} S_1(t)B$ is a relatively compact set in $\mathscr{K}$. Then, by applying [19, p. 23, Theorem 1.1], we can show the existence of the global attractor for the partly dissipative system of the Deneubourg model:

Theorem 4. *The ω-limit set of $\mathscr{B}$, $\mathscr{A} = \bigcap_{0 \le t < \infty} \overline{\bigcup_{t \le s < \infty} S_1(t)\mathscr{B}}$, is the global attractor for the dynamical system $(S(t), \mathscr{K}, \mathscr{H})$. In addition, the global attractor $\mathscr{A}$ is connected in $\mathscr{K}$.*

Let us introduce a positively invariant set

$$\mathscr{X} = \bigcup_{t \ge t_{\mathscr{B}}} S(t)\mathscr{B} \subset \mathscr{B}.$$

The asymptotic behavior of the solutions thereby reduces to the eventual dynamical system $(S(t), \mathscr{X}, \mathscr{H})$. From the existence of a compact absorbing set in $\mathscr{H}$ (Theorem 3), there exists a uniform constant K_r for $\|u(t)\|_{\mathscr{C}}$, of order $O(1)$ for large μ, that is,

$$\|u(t)\|_{\mathscr{C}} \le C\|U(t)\|_{H^1 \times L_2 \times H^2} \le C \cdot r := K_r$$
$$\text{for all } U(t) = {}^T[u(t) \;\; v(t) \;\; w(t)] \in \mathscr{X}, \tag{30}$$

where C is an embedding constant of (5) and r is the radius of absorbing ball $\mathscr{B}$.

In addition, with suitably large μ, we can construct a global Lyapunov functional for the unique constant equilibrium:

$$U^* = {}^T[u^* \;\; v^* \;\; w^*] := {}^T\left[\frac{1}{\mu} \;\; \frac{1}{\mu} \;\; \frac{1}{\mu}\right]. \tag{31}$$

To construct the Lyapunov functional, the uniform boundedness (30) of the maximum norm of u plays a crucial role. We then show the following:

Theorem 5. Assume that $\mu > \chi\sqrt{K_r}/4$. Here, K_r is the upper bound defined in (30). Then, a functional on $\mathscr{X}$

$$\Phi(U(t)) = \int_\Omega \left[\mu u - 1 - \log \mu u + \frac{\delta\mu^2}{K_r}(v - v^*)^2 + \frac{\tau\chi^2}{8}(w - w^*)^2\right] dx$$

satisfies $\frac{d}{dt}\Phi(U(t)) \le 0$, $\Phi(U) > 0$ $(U \ne U^*)$, and $\Phi(U^*) = 0$.

REMARK 1. *Because* $K_r = O(1)$ *for large* μ, *the region of* (χ, μ) *contained in* $\mathbb{R}_+^2$ *satisfying the inequality* $\mu > \chi\sqrt{K_r}/4$ *is non-empty.*

PROOF. It is clear that $\Phi(U) > 0$ $(U \neq U^*)$ and $\Phi(U^*) = 0$. We can show $\frac{d}{dt}\Phi(U(t)) \leq 0$ in a similar manner to [4, 11], except for the need to construct an L_2-absorbing $\|u - u^*\|_{L_2}^2$. By noting $\|u(t)\|_{\mathscr{C}} \leq K_r$, we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (\mu u - \log \mu u)dx &= \int_\Omega \left(\mu - \frac{1}{u}\right)[(u_x - \chi u w_x)_x + (1 - \mu u)]dx \\
&= -\int_\Omega \frac{\partial}{\partial x}\left(\mu - \frac{1}{u}\right)(u_x - \chi u w_x)dx - \int_\Omega \frac{1}{u}(1 - \mu u)^2 dx \\
&\leq -\int_\Omega \frac{u_x^2}{u^2}\, dx + \int_\Omega \chi \frac{u_x w_x}{u}\, dx - \frac{\mu^2}{K_r}\int_\Omega (u - u^*)^2 dx.
\end{aligned}$$

Similarly, we have

$$\frac{d}{dt}\int_\Omega \frac{\delta\mu^2}{K_r}(v - v^*)^2 dx = -\frac{2\mu^2}{K_r}\int_\Omega (v - v^*)^2 dx + \frac{2\mu^2}{K_r}\int_\Omega (v - v^*)(u - u^*)dx,$$

and also

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{\tau\chi^2}{8}(w - w^*)^2 dx &= \int_\Omega (w - w^*)(\tau w_t)dx = \int_\Omega (w - w^*)(w_{xx} + v - w)dx \\
&= -\frac{\chi^2}{4}\int_\Omega w_x^2\, dx + \frac{\chi^2}{4}\int_\Omega (v - v^*)(w - w^*)dx \\
&\quad -\frac{\chi^2}{4}\int_\Omega (w - w^*)^2 dx.
\end{aligned}$$

It is concluded that

$$\begin{aligned}
\frac{d}{dt}\Phi(U(t)) &= \frac{d}{dt}\int_\Omega \left[\mu u - 1 - \log \mu u + \delta\mu^2(v - v^*)^2 + \frac{\tau\chi^2}{8}(w - w^*)^2\right]dx \\
&\leq -\int_\Omega \frac{u_x^2}{u^2}\, dx + \int_\Omega \chi \frac{u_x w_x}{u}\, dx - \frac{\chi^2}{4}\int_\Omega w_x^2\, dx - \frac{\mu^2}{K_r}\int_\Omega (u - u^*)^2 dx \\
&\quad + \frac{2\mu^2}{K_r}\int_\Omega (v - v^*)(u - u^*)dx - \frac{2\mu^2}{K_r}\int_\Omega (v - v^*)^2 dx \\
&\quad - \frac{\chi^2}{4}\int_\Omega (w - w^*)^2 dx + \frac{\chi^2}{4}\int_\Omega (v - v^*)(w - w^*)dx
\end{aligned}$$

$$= -\int_\Omega \left(\frac{u_x}{u} - \frac{\chi}{2} w_x\right)^2 dx - \frac{\mu^2}{K_r}\int_\Omega [(u-u^*) - (v-v^*)]^2 dx$$

$$-\frac{\mu^2}{K_r}\int_\Omega \left[(v-v^*) - \frac{\chi^2 K_r}{8\mu^2}(w-w^*)\right]^2 dx$$

$$-\frac{\chi^2}{4}\left(1 - \frac{\chi^2 K_r}{16\mu^2}\right)\int_\Omega (w-w^*)^2 dx.$$

Therefore, we have $\frac{d}{dt}\Phi(U(t)) \leq 0$ under the condition $\mu > \chi\sqrt{K_r}/4$. □

Finally, we show the convergence of U to U^* with the maximum norm.

Proposition 7. *Under the condition $\mu > \chi\sqrt{K_r}/4$, the convergence of $U(t)$ to U^* is uniform for u and w:*

$$\|u(t) - u^*\|_{\mathscr{C}} \to 0, \qquad \|v(t) - v^*\|_{L_2} \to 0, \qquad \|w(t) - w^*\|_{\mathscr{C}^1} \to 0, \qquad t \to \infty.$$

Proof. By referring to, e.g., [1], we can show the convergence. From the proof of Theorem 5, we have

$$\frac{d}{dt}\Phi(U(t)) \leq -\eta\int_\Omega \left[[(u-u^*) - (v-v^*)]^2 + \left[(v-v^*) - \frac{\chi^2 K_r}{8\mu^2}(w-w^*)\right]^2 + (w-w^*)^2\right] dx,$$

where $\eta := \min\left\{\frac{\mu^2}{K_r}, \frac{\chi^2}{4}\left(1 - \frac{\chi^2 K_r}{16\mu^2}\right)\right\}$. We set $\varphi(t) := \int_\Omega \left[[(u-u^*) - (v-v^*)]^2 + \left[(v-v^*) - \frac{\chi^2 K_r}{8\mu^2}(w-w^*)\right]^2 + (w-w^*)^2\right] dx$. Then, by integrating (32) from 1 to t, we have $\int_1^\infty \varphi(s)ds \leq \frac{1}{\eta}\Phi(U(1)) < \infty$. The positivity of $\varphi(t)$ indicates that $\varphi(t) \to 0$ $(t \to \infty)$. We then have the convergence to the constant solution U^* in L_2-norm. Since the solution U belongs to the functional space (2), the convergences in $\mathscr{K}$ and in the maximum norm are proved from the Gagliardo-Nirenberg inequality, e.g., $\|u\|_{\mathscr{C}} \leq C\|u\|_{H^{3/4}} \leq C\|u\|_{H^1}^{3/4}\|u\|_{L_2}^{1/4}$ and $\|w\|_{\mathscr{C}^1} \leq C\|w\|_{H^{7/4}} \leq C\|w\|_{H^2}^{7/8}\|w\|_{L_2}^{1/8}$. □

Remark 2. *The linearized analysis* [13] *shows that if $\chi > \mu(\sqrt{\mu}+1)^2$, then the constant equilibrium is unstable, and, conversely, that if $\chi < \mu(\sqrt{\mu}+1)^2$, then the constant equilibrium is locally asymptotically stable. This implies that the region (χ, μ) specified by the assumption of Theorem* 5 *and Proposition* 7*, or equivalently, $\chi < 4\mu/\sqrt{K_r}$, should be included in the locally stable region of $\chi < \mu(\sqrt{\mu}+1)^2$. This shows that the upper bound K_r in* (30) *can be estimated from below: $K_r \geq 16$, even if either χ or $1/\mu$ is sufficiently small.*

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