Function of Self-Adjoint Transformation.

By

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Introduction.

Let H be a self-adjoint transformation in Hilbert space $\mathfrak{H}$, $E(U)$ be the corresponding resolution of the identity in the sense of F. Maeda[(1)] which depends on Borel set U of 1-dimensional Euclidean space R_1, and $f(\lambda)$ be any Baire function defined in R_1. If we put

$$F(H) = \int_{R_1} f(\lambda) E(dU),$$

then $F(H)$ is a closed linear transformation,[(2)] which we call a function of H.

J. v. Neumann[(3)] and F. Riesz[(4)] have shown the theorem equivalent to the following: *When H is a bounded self-adjoint transformation, then a bounded linear transformation A is a function of H, if and only if A is permutable with any bounded linear transformation which is permutable with H.* And F. Riesz has remarked that anologous theorems hold for the unbounded cases.

In this paper we shall show that any transformation A defined on a subset of $\mathfrak{H}$ is a contraction of a function of H when A is permutable with any bounded linear transformation[(5)] permutable with H.[(6)] Moreover A is a function of H if and only if A is a closed linear trans-

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(1) F. Maeda, this journal, **4** (1934), 78.

(2) F. Maeda, *ibid.* 85–88.

(3) J. v. Neumann, Math. Ann. **102** (1929), 370–427. Annals of Math., **32** (1931), 191–226.

(4) F. Riesz, Acta Szeged, **7** (1935), 147–159.

(5) In the following proofs it is sufficient to consider only bounded linear transformations with norm $\leqq 1$.

(6) Prof. Y. Mimura of Osaka University reported a similar result in the annual meeting of the Physico-Mathematical Society held at Tokyo on April, 2, 1936.

formation with domain dense in $\mathfrak{H}$. Next we consider the case when H has a simple spectrum.

We take this opportunity of thanking Prof. F. Maeda for his kind guidance.

In this paper we shall use the following notations:

H ....Self-adjoint transformation.
$E(U)$ ....Resolution of the identity corresponding to H.
A ....Transformation defined on a subset of $\mathfrak{H}$ and permutable with any bounded linear transformation permutable with H.(1)
P ....Projective transformation.
$\mathfrak{M}_{(\mathfrak{b})}$ ....Closed linear manifold determined by the system $\{E(U)\mathfrak{b}\}$, U being a parameter.
$\mathfrak{D}_T$ ....Domain of the transformation T.
$T \subset S$ ....T is a contraction of S.
$\overbrace{\mathfrak{H}\times\mathfrak{H}\times\cdots\times\mathfrak{H}}^{n}$ ....Cartesian product(2) of $\mathfrak{H}$.
$\mathfrak{f}, [\mathfrak{f}_1, \mathfrak{f}_2, \cdots, \mathfrak{f}_n]$ ...Element of $\mathfrak{H}\times\mathfrak{H}\times\cdots\times\mathfrak{H}$.
$\boldsymbol{T}, [T, T, \cdots, T]$ ...Transformation in $\mathfrak{H}\times\mathfrak{H}\times\cdots\times\mathfrak{H}$ which transforms $[\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n]$ into $[T\mathfrak{f}_1, T\mathfrak{f}_2, \ldots, T\mathfrak{f}_n]$

1. Let P be the projection on the closed linear manifold $\mathfrak{M}_{(\mathfrak{b})}$, then

$$P\mathfrak{f} = \int_{R_1} \frac{(\mathfrak{f}, E(dU)\mathfrak{b})}{\| E(dU)\mathfrak{b} \|^2} E(dU)\mathfrak{b} \,.^{(3)}$$

Hence

(1) We do not suppose A to be linear. That A is permutable with a bounded linear transformation is meant by $BA \subset AB$.

(2) In this space the fundamental operations are defined as follows:

$$[\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n]+[\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n] = [\mathfrak{f}_1+\mathfrak{g}_1, \mathfrak{f}_2+\mathfrak{g}_2, \ldots, \mathfrak{f}_n+\mathfrak{g}_n]$$

$$c[\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n] = [c\mathfrak{f}_1, c\mathfrak{f}_2, \ldots, c\mathfrak{f}_n] \qquad c: \text{complex number.}$$

$$([\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n], [\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n]) = (\mathfrak{f}_1, \mathfrak{g}_1)+(\mathfrak{f}_2, \mathfrak{g}_2)+ \ldots +(\mathfrak{f}_n, \mathfrak{g}_n).$$

Then this is also a Hilbert space. Cf. M. H. Stone, Linear transformation in Hilbert space, (1932), 30.

(3) Put $\mathfrak{f} = \mathfrak{f}_1+\mathfrak{f}_2$, where $\mathfrak{f}_1 = P\mathfrak{f}$. Then $\mathfrak{f}_1 = \int_{R_1} \frac{(\mathfrak{f}_1, E(dU)\mathfrak{b})}{\| E(dU)\mathfrak{b} \|^2} E(dU)\mathfrak{b}$. But since $\mathfrak{f}_2$ is perpendicular to $\mathfrak{M}_{(\mathfrak{b})}$, $(\mathfrak{f}_2, E(U)\mathfrak{b}) = 0$ for all U. Hence $(\mathfrak{f}_1, E(U)\mathfrak{b}) = (\mathfrak{f}, E(U)\mathfrak{b})$. Cf. F. Maeda, **4** (1934), 73; and **6** (1936), 36.

$$PE(V)\mathfrak{f} = \int_{R_1} \frac{\big(E(V)\mathfrak{f}, E(dU)\mathfrak{b}\big)}{\| E(dU)\mathfrak{b} \|^2} E(dU)\mathfrak{b} = \int_{R_1} \frac{\big(\mathfrak{f}, E(VdU)\mathfrak{b}\big)}{\| E(dU)\mathfrak{b} \|^2} E(dU)\mathfrak{b}$$

$$= \int_{V} \frac{\big(\mathfrak{f}, E(dU)\mathfrak{b}\big)}{\| E(dU)\mathfrak{b} \|^2} E(dU)\mathfrak{b} = E(V)P\mathfrak{f}.^{(1)}$$

From this fact we can see that P is permutable with H,[(2)] hence P is permutable with A. It is also the case for $1-P$. Therefore if $\mathfrak{f} \varepsilon \mathfrak{D}_A$, then its projections on $\mathfrak{M}_{(\mathfrak{b})}$ and $\mathfrak{H} \ominus \mathfrak{M}_{(\mathfrak{b})}$ also belong to $\mathfrak{D}_A$.

2. Since $\mathfrak{H}$ is separable there exists a sequence $\{\mathfrak{h}_i\}$ which is dense in $\mathfrak{H}$. Put $\mathfrak{b}_1 = \mathfrak{h}_1$, and construct $\mathfrak{M}_{(\mathfrak{b}_1)}$. Let $\mathfrak{h}_2'$ be the first element of $\{\mathfrak{h}_i\}$ which is not contained in $\mathfrak{M}_{(\mathfrak{b}_1)}$, then its projection on $\mathfrak{H} \ominus \mathfrak{M}_{(\mathfrak{b}_1)}$ belongs to $\mathfrak{D}_A$, which is denoted by $\mathfrak{b}_2$. Let $\mathfrak{h}_3'$ be the first element of $\{\mathfrak{h}_i\}$, which is not contained in $\mathfrak{M}_{(\mathfrak{b}_1)} \oplus \mathfrak{M}_{(\mathfrak{b}_2)}$, and its projection on $\mathfrak{H} \ominus \mathfrak{M}_{(\mathfrak{b}_1)} \oplus \mathfrak{M}_{(\mathfrak{b}_2)}$, be denoted by $\mathfrak{b}_3$, then $\mathfrak{b}_3 \varepsilon \mathfrak{D}_A$. We construct $\mathfrak{M}_{(\mathfrak{b}_3)}$ as above and continue this process indefinitely. Then we have at most denumerably infinite $\mathfrak{M}_{(\mathfrak{b}_i)}$. Then for any element $\mathfrak{g}$ of $\mathfrak{D}_A$,

$$\mathfrak{g} \varepsilon \mathfrak{M}_{(\mathfrak{b}_1)} \oplus \mathfrak{M}_{(\mathfrak{b}_2)} \oplus \dots,$$

hence we have

$$\mathfrak{g} = \sum_i \int_{R_1} \frac{\big(\mathfrak{g}, E(dU)\mathfrak{b}_i\big)}{\| E(dU)\mathfrak{b}_i \|^2} E(dU)\mathfrak{b}_i$$

Therefore if for some set V $E(V)\mathfrak{b}_i = 0$, $i = 1, 2, \dots$, then we have $E(V)\mathfrak{g} = 0$.

3. We consider a Hilbert space $\mathfrak{H} \times \mathfrak{H} \times \cdots \times \mathfrak{H}$. Let $E(U)$ be $[E(U), \dots, E(U)]$, and $\mathfrak{f} = [\mathfrak{f}_1, \dots, \mathfrak{f}_n]$ be any element of this space. Then we have

(α) $$E(U)E(U')\mathfrak{f} = [E(U)E(U')\mathfrak{f}_1, \dots] = [E(UU')\mathfrak{f}_1, \dots] = E(UU')\mathfrak{f}.$$

(β) When $$U = \sum_i U_i,$$

$$\| E(U)\mathfrak{f} - \sum_{i=1}^{N} E(U_i)\mathfrak{f} \|^2 = \| E(\sum_{j=N+1}^{\infty} U_i)\mathfrak{f} \|^2 = \sum_{j=1}^{n} \| E(\sum_{i=N+1}^{\infty} U_i)\mathfrak{f}_j \|^2.$$

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(1) F. Maeda, this volume, 38.

(2) For bounded linear transformation is permutable with H if it is permutable with $E(U)$.

Since $\| E(\sum_{i=N+1}^{\infty} U_i)\mathfrak{f}_j \|^2$ tends to zero with $\frac{1}{N}$, we have

$$E(U)\mathfrak{f} = \sum_{i=1}^{\infty} E(U_i)\mathfrak{f}.$$

$$(\gamma) \qquad E(R_1)\mathfrak{f} = [E(R_1)\mathfrak{f}_1, \ldots\ldots] = [\mathfrak{f}, \ldots\ldots] = \mathfrak{f}.$$

$$(\delta) \quad \big(E(U)\mathfrak{f}, \mathfrak{g}\big) = \sum_{i=1}^{n} \big(E(U)\mathfrak{f}_i, \mathfrak{g}_i\big) = \sum_{i=1}^{n} \big(\mathfrak{f}_i, E(U)\mathfrak{g}_i\big) = \big(\mathfrak{f}, E(U)\mathfrak{g}\big).$$

Hence $E(U)$ *is a resolution of the identity.*[1]

4. Let P be the projection on the manifold $\mathfrak{M}_{(\mathfrak{f})}$ in $\mathfrak{H}\times\cdots\times\mathfrak{H}$, and $A = [A, A, \ldots\ldots, A]$. If $\mathfrak{f} \varepsilon \mathfrak{D}_A$, then

$$PA\mathfrak{f} = \int_{R_1} \frac{\big(A\mathfrak{f}, E(dU)\mathfrak{f}\big)}{\| E(dU)\mathfrak{f} \|^2} E(dU)\mathfrak{f}$$

$$= \left[\int_{R_1} \frac{\sum_i \big(A\mathfrak{f}_i, E(dU)\mathfrak{f}_i\big)}{\sum_i \| E(dU)\mathfrak{f}_i \|^2} E(dU)\mathfrak{f}_1, \ldots\ldots\right]$$

$$= \left[\sum_j \int_{R_1} \frac{\big(A\mathfrak{f}_j, E(dU)\mathfrak{f}_j\big)}{\sum_i \| E(dU)\mathfrak{f}_i \|^2} E(dU)\mathfrak{f}_1, \ldots\ldots\right]$$

$$= \left[\sum_j A \int_{R_1} \frac{\big(\mathfrak{f}_i, E(dU)\mathfrak{f}_j\big)}{\sum_i \| E(dU)\mathfrak{f}_i \|^2} E(dU)\overset{(2)}{\mathfrak{f}_1}, \ldots\ldots\right]$$

$$= \left[\sum_j \int_{R_1} \frac{\| E(dU)\mathfrak{f}_j \|^2}{\sum_i \| E(dU)\mathfrak{f}_i \|^2} E(dU)\overset{(3)}{A}\mathfrak{f}_1 \ldots\ldots\right]$$

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(1) F. Maeda, this journal, **4** (1934), 78.

(2) Transformation $B\mathfrak{x} = \int_{R_1} \frac{(\mathfrak{x}, E(dU)\mathfrak{f}_j)}{\sum_i \| E(dU)\mathfrak{f}_i \|^2} E(dU)\mathfrak{f}_1$ is a bounded linear transformation and permutable with $E(V)$, V being any Borel set in R_1. For $\| B\mathfrak{x} \|^2 = \int_{R_1} \frac{|(\mathfrak{x}, E(dU)\mathfrak{f}_j)|^2}{(\sum_i \| E(dU)\mathfrak{f}_i \|^2)^2} \| E(dU)\mathfrak{f}_1 \|^2 \leqq \int_{R_1} \frac{\| E(dU)\mathfrak{x} \|^2 \; \| E(dU)\mathfrak{f}_j \|^2 \; \| E(dU)\mathfrak{f}_1 \|^2}{(\sum_i \| \mathfrak{x}(dU)\mathfrak{f}_i \|^2)^2}$ $\leqq \int_{R_1} \| E(dU)\mathfrak{x} \|^2 = \| \mathfrak{x} \|^2$. Linearity of B is obvious and, as in sec. 1. $E(V)B = BE(V)$. Hence by the assumption A and B are permutable.

(3) Transformation $C\mathfrak{x} = \int_{R_1} \frac{\| E(dU)\mathfrak{f}_i \|^2}{\sum_i \| E(dU)\mathfrak{f}_i \|^2} E(dU)\mathfrak{x}$ has the same properties as B in footnote (2), the proof being analogous to footnote (2).

$$= \left[\int_{R_1} E(dU)A\mathfrak{f}_1, \ldots\ldots\right]$$

$$= [A\mathfrak{f}_1, \ldots\ldots, A\mathfrak{f}_n].$$

On the other hand we can write

$$PA\mathfrak{f} = \int_{R_1} F_n(\lambda)E(dU)\mathfrak{f}^{(1)}$$

$$= \left[\int_{R_1} F_n(\lambda)E(dU)\mathfrak{f}_1, \ldots\ldots\right]$$

Therefore

$$A\mathfrak{f}_i = \int_{R_1} F_n(\lambda)E(dU) \qquad i = 1, 2, \ldots\ldots, n.$$

Let $\{\mathfrak{f}_i\}$ be a sequence belonging to $\mathfrak{D}_A$. From the fact mentioned above, we have

$$A\mathfrak{f}_i = \int_{R_1} F_n(\lambda)E(dU)\mathfrak{f}_i \qquad \begin{array}{l} i = 1, 2, \ldots\ldots \\ n = i,\ i+1, \ldots\ldots \end{array}$$

Let the set of the points λ for which $F_n(\lambda) \neq F_m(\lambda)$ for some m, n be denoted by V, then

$$E(V)\mathfrak{f}_i = 0^{(2)} \qquad i = 1, 2, \ldots\ldots$$

If we put

$$F(\lambda) = \overline{\lim_{n\to\infty}}\, \Re\big(F_n(\lambda)\big) + i\, \overline{\lim_{n\to\infty}}\, \Im\big(F_n(\lambda)\big)^{(3)} \quad \text{if the limits are finite,}$$

$$= 0 \qquad \text{otherwise,}$$

then we have

$$A\mathfrak{f}_i = \int_{R_1} F(\lambda)E(dU)\mathfrak{f}_i, \qquad i = 1, 2, \ldots\ldots$$

5. Consider the sequence $\{\mathfrak{b}_i\}$ defined in sec. 2., and any $\mathfrak{g} \in \mathfrak{D}_A$, then by preceding section for $\{\mathfrak{b}_i\}$ and $\{\mathfrak{g}, \mathfrak{b}_1, \mathfrak{b}_2, \ldots\ldots\}$ we have two functions $F(\lambda)$ and $F^*(\lambda)$ such that:

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(1) F. Maeda, *loc. cit.* 73. cf. sec. 1. footnote (1).

(2) Since $A\mathfrak{f}_i = \int_{R_1} F_n(\lambda)E(dU)\mathfrak{f}_i = \int_{R_1} F_m(\lambda)E(dU)\mathfrak{f}_i$, we have $\int_{R_1} | F_n(\lambda) - F_m(\lambda) |^2 \, || E(dU)\mathfrak{f}_i ||^2 = 0$. Therefore $E(U) = 0$.

(3) Real and imaginary parts of $F_n(\lambda)$ are respectively denoted by $\Re(F_n(\lambda))$ and $\Im(F_n(\lambda))$.

$$(1) \qquad A\mathfrak{b}_i = \int_{R_1} F(\lambda)E(dU)\mathfrak{b}_i \qquad i = 1, 2, \ldots .$$

$$A\mathfrak{b}_i = \int_{R_1} F^*(\lambda)E(dU)\mathfrak{b}_i \qquad i = 1, 2, \ldots .$$

$$(2) \qquad A\mathfrak{g} = \int_{R_1} F^*(\lambda)E(dU)\mathfrak{g} .$$

Let V be the set of the points λ for which $F(\lambda) \neq F^*(\lambda)$, then from (1) we are led to

$$E(V)\mathfrak{b}_i = 0 \qquad i = 1, 2, \ldots .$$

therefore from sec. 2., we get

$$E(V)\mathfrak{g} = 0 ,$$

hence

$$A\mathfrak{g} = \int_{R_1} F(\lambda)E(dU)\,\mathfrak{g} .$$

which completes the proof of the theorem mentioned in the introduction.

6. If moreover A is closed and $\mathfrak{D}_A$ is linear and dense in $\mathfrak{H}$, then A is *a function of* H.

For by preceding section,

$$(1) \qquad A \subset F(H) = \int_{R_1} E(\lambda)E(aU)$$

hence

$$(2) \qquad A^* \supset F^*(H) = \int_{R_1} \overline{F(\lambda)}E(aU)$$

Therefore

$$\mathfrak{D}_A \subset \mathfrak{D}_{F(H)} = \mathfrak{D}_{F^*(H)} \subset \mathfrak{D}_{A^*} .$$

that is,

$$\mathfrak{D}_A \subset \mathfrak{D}_{A^*}$$

If we consider A^* instead of A, then we have

$$\mathfrak{D}_{A^*} \subset \mathfrak{D}_{A^{**}}$$

Since $A = A^{**}$ [(1)] we have

$$\mathfrak{D}_A = \mathfrak{D}_{F(H)}$$

hence

$$A \equiv F(H) .$$

(1) From (1) and (2) A becomes a $**$-transformation. Cf. J. v. Neumann, Annales of Math, *loc. cit*, 301.

And that these conditions are necessary is easily seen.

7. Next we shall consider when is the case that any bounded linear transformation B permutable with H is *a function of H.*[1] As in sec. 2. we construct the closed linear manifolds $\mathfrak{M}_{(\mathfrak{b}_i)}$ starting from a dense set in $\mathfrak{H}$ such that

$$\mathfrak{H} = \mathfrak{M}_{(\mathfrak{b}_1)} \oplus \mathfrak{M}_{(\mathfrak{b}_2)} \oplus \dots .$$

Let $\{a_i\}$ be a sequence of positive numbers such that $\sum a_i \mathfrak{b}_i$ converges. Put

$$\mathfrak{b} = \sum a_i \mathfrak{b}_i .$$

If for some Borel set V, $E(V)\mathfrak{b} = 0$, then since

$$E(V)\mathfrak{b} = \sum E(V) a_i \mathfrak{b}_i$$

that is

$$\| E(V)\mathfrak{b} \|^2 = \sum a_i^2 \| E(V)\mathfrak{b}_i \|^2$$

hence we have

$$E(V)\mathfrak{b}_i = 0 , \qquad i = 1, 2, \dots .$$

Therefore for any element $\mathfrak{g}$ of $\mathfrak{H}$, as in sec. 2., we have $E(V)\mathfrak{g} = 0$

Consider the projection P on the manifold $\mathfrak{M}_{(\mathfrak{b})}$. Since P is permutable with H, by the assumption P is a function of H. Hence we can write

$$(1) \qquad P = \int_{R_1} p(\lambda) E(dU) .$$

From the definition of the projection

$$\mathfrak{b} = \int_{R_1} p(\lambda) E(dU)\mathfrak{b}$$

therefore $P(\lambda) = 1$ except the set V for which $E(V)\mathfrak{b} = 0$. Hence from above $E(V)\mathfrak{g} = 0$ for any $\mathfrak{g}$. Then, since we can write 1 instead of $p(\lambda)$ in (1), P becomes the identical transformation. Therefore $\mathfrak{H} = \mathfrak{M}_{(\mathfrak{b})}$, that is, *H has a simple spectrum.*

Conversely if *H has a simple spectrum*, then there exists a element $\mathfrak{b}$ such that $\mathfrak{H} = \mathfrak{M}_{(\mathfrak{b})}$, and any $\mathfrak{g}$ can be written in the form

$$\mathfrak{g} = \int_{R_1} g(\lambda) E(dU)\mathfrak{b} .$$

Let B be permutable with H, then

(1) M. H. Stone, *loc. cit.* 300.

$$B\mathfrak{g} = \int_{R_1} g(\lambda)E(dU)B\mathfrak{b}\,.$$

Since $B\mathfrak{b} \varepsilon \mathfrak{M}_{(\mathfrak{b})}$, we can write $B\mathfrak{b} = \int_{R_1} \beta(\lambda)E(dU)\mathfrak{b}$. Therefore we have

$$B\mathfrak{g} = \int_{R_1} g(\lambda)\beta(\lambda)E(dU)\mathfrak{b}\ ^{(1)}$$

$$= \int_{R_1} \beta(\lambda)E(dU)\int_{R_1} g(\lambda)E(dU)\mathfrak{b}\ ^{(2)}$$

$$= \int_{R_1} \beta(\lambda)E(dU)\mathfrak{g}\,.$$

Hence B is a function of H.

Thus we have proved the two following statesments are equivalent:

(α) *H has a simple spectrum.*

(β) *Any bounded linear transformation permutable with H is a function of H.*

8. In this section we consider the case when H has a simple spectrum. If A is permutable with any bounded function of H, A is a contraction of a function of H. For A is permutable with any bounded linear transformation permutable with H by sec. 7.

Let A be a closed linear transformation permutable with $E(U)$, where $\mathfrak{D}_A$ may not be dense in $\mathfrak{H}$. Then, since A is permutable with any bounded function of H,[3] *A is a contraction of a function of H.*

(1) F. Maeda, this journal, **4** (1934), 76. T. Ogasawara, this volume, 54.
(2) T. Ogasawara, *ibid.*
(3) F. Maeda, *ibid.* 63.