Representations of Linear Operators by Differential Set Functions.

By

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Dirac considered, in his treatise on quantum mechanics,[1] representations of states and observables, which may be described with slight modifications as follows: Let $\mathfrak{H}$ be a linear vector space, where the inner products are defined. Each vector in $\mathfrak{H}$ means a state of a dynamical system, and self-adjoint operators mean the observables. Let $\{\mathfrak{g}_r\}$ be a complete normalized orthogonal system in $\mathfrak{H}$ so that

$$(\mathfrak{g}_r, \mathfrak{g}_s) = \delta_{rs}, \qquad (1)$$

where the sysmbol δ_{rs} has the meaning

$$\delta_{rs} = 0 \quad \text{when} \quad r \neq s$$
$$= 1 \quad \text{when} \quad r = s.$$

If we take $\{\mathfrak{g}_r\}$ for a coordinate system, the representative of any vector $\mathfrak{f}$ is the coordinates $\{a_r\}$ of $\mathfrak{f}$ so that

$$a_r = (\mathfrak{f}, \mathfrak{g}_r) \qquad (r = 1, 2, \ldots.),$$

and the representative of a self-adjoint operator $\boldsymbol{H}$ is a matrix a_{rs} so that

$$a_{rs} = (\boldsymbol{H}\mathfrak{g}_s, \mathfrak{g}_r) \qquad (r, s = 1, 2, \ldots.).$$

And the representative of $\boldsymbol{H}\mathfrak{f}$ is $\{b_r\}$ so that

$$b_r = \sum_s a_{rs}a_s \qquad (r = 1, 2, \ldots.).$$

It is convenient in treating of physical problems, to use a representation in which the elements of the coordinate system $\{\mathfrak{g}_r\}$ are

(1) P. A. M. Dirac, *The Principles of Quantum Mechanics*, sec. ed. (1935), 49–87.

eigenvectors of a self-adjoint operator. But the above method is applicable only in the case when the chosen self-adjoint operator has as eigenvalues a discrete set of numbers. To extend the above method to the case when the chosen self-adjoint operator has as eigenvalues all numbers in a certain range, Dirac introduced the improper δ function, defined by

$$\int_{-\infty}^{+\infty} \delta(x)dx = 1$$

$$\delta(x) = 0 \qquad \text{for} \qquad x \neq 0 .$$

In this case, the coordinate system $\{\mathfrak{g}_r\}$ has a continuous parameter and it satisfies the following condition

$$(\mathfrak{g}_r, \mathfrak{g}_s) = \delta(r-s) \tag{2}$$

which corresponds to (1). The representative of any vector $\mathfrak{f}$ is a point function $f(r)$

$$f(r) = (\mathfrak{f}, \mathfrak{g}_r) ,$$

and the representative of a self-adjoint operator $\boldsymbol{H}$ is $K(r, s)$ so that

$$K(r, s) = (\boldsymbol{H}\mathfrak{g}_s, \mathfrak{g}_r) . \tag{3}$$

And $\boldsymbol{H}\mathfrak{f}$ is represented by an integrating operator

$$\int_{-\infty}^{+\infty} K(r, s)f(s)ds . \tag{4}$$

Thus the whole representation theory for a discrete set of eigenvalues may be taken over to the case of a continuous range of eigenvalues. We simply have to replace sums by integrals and the two-suffix δ symbol δ_{rs} by the δ function $\delta(r-s)$ all the way through.

But the definition and the use of the improper δ function are not rigorous according to the standards of pure mathematics. An alternating way of defining the δ function is as the differential coefficient $\epsilon'(x)$ of the function $\epsilon(x)$ given by

$$\begin{aligned} \epsilon(x) &= 0 \qquad \text{when} \qquad x < 0 \\ &= 1 \qquad \text{when} \qquad x > 0 . \end{aligned} \tag{5}$$[1]

(1) Dirac, loc. cit., 74.

Hence, if we can use $\epsilon(x)$ or a corresponding function of it, we can eliminate the improper function. This purpose can be attained when we use the differential set functions, which I have investigated in my previous paper.(1)

Let U and U' be intervals in the space of real numbers. Corresponding to (5), we define $\epsilon_U(x)$ as follows:

$$\begin{aligned} \epsilon_U(x) &= 0 \qquad \text{when } x \text{ is not in } U \\ &= 1 \qquad \text{when } x \text{ is in } U. \end{aligned}$$

Then the differential set function which corresponds to $\epsilon_U(x)$ is

$$\int_{U'} \epsilon_U(x)dx = \beta(UU')$$

where $\beta(U)$ is the length of the interval U. $\beta(UU')$ has the same property as the improper δ function. For example,

$$\int_{R_1} \frac{\beta(U \cdot dU')\varphi(dU')}{\beta(dU')} = \varphi(U)$$

corresponds to

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)dx = f(a).^{(2)}$$

I have seen in my previous paper,(3) the eigenvectors of a self-adjoint operator $\boldsymbol{H}$ are expressed by a vector valued differential set function $\mathfrak{q}(U)$ irrespective of whether $\boldsymbol{H}$ has a discrete spectrum or a continuous spectrum. And $\mathfrak{q}(U)$ satisfies the following condition

$$\big(\mathfrak{q}(U), \mathfrak{q}(U')\big) = \sigma(UU')$$

which corresponds to (2). When $\{\mathfrak{q}(U)\}$ is a complete set of eigenvectors of $\boldsymbol{H}$, U being a parameter, the representative of any vector $\mathfrak{f}$ is the differential set function

$$\xi(U) = \big(\mathfrak{f}, \mathfrak{q}(U)\big),$$

which may be considered as a coefficient of the expansion of $\mathfrak{f}$ with

(1) F. Maeda, "Space of Differential Set Funktions," this volume, 19-34.
(2) Dirac, loc. cit., 73.
(3) F. Maeda, ibid., 39-40.

respect to the generalized normalized orthogonal system $\{\mathfrak{q}(U)\}$, about which I have investigated in detail in my previous papers.[1]

To represent the self-adjoint operator $\boldsymbol{H}$, we may use Dirac's method (3), so that

$$\mathfrak{K}(U, U') = \left(\boldsymbol{H}\mathfrak{q}(U'), \mathfrak{q}(U)\right),$$

and $\boldsymbol{H}\mathfrak{f}$ may be represented by an integrating operator

$$\int_{R_1} \frac{\mathfrak{K}(U, dU')\xi(dU')}{\sigma(dU')}$$

which corresponds to (4).

Thus we can represent vectors and operators by differential set functions, without using Dirac's improper function. Moreover, our method implies both the cases of discrete eigenvalues and continuous eigenvalues.

In this paper, using the differential set functions, I investigate the representation theory of linear operators, and next I apply this theory to the self-adjoint operators, and I give the physical interpretation of the representatives as the probability amplitude. In the last part of this paper I deduce one form of the principle of uncertainty from the representation theory for the coordinate and its conjugate momentum.

Representations of Linear Operators in the Simple Case.

1. Let $\mathfrak{H}$ be a space of vectors, which satisfies the following axioms:

(i) $\mathfrak{H}$ is a linear space.

(ii) In $\mathfrak{H}$ an inner product is defined.

(iii) $\mathfrak{H}$ is complete.

And let $\mathfrak{q}(U)$ be a completely additive vector valued differential set function defined for all sets of a differential set system in an abstract space V.[2] Then $\sigma(U) = \|\mathfrak{q}(U)\|^2$ is a completely additive non-negative differential set function in V, and

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(1) F. Maeda, this journal, **4** (1934), 69–75; this volume, 36–37.

(2) Cf. F. Maeda, this volume, 33.

$$\big(\mathfrak{q}(U), \mathfrak{q}(U')\big) = \sigma(UU').^{(1)} \tag{1.1}$$

Hence, we may consider $\{\mathfrak{q}(U)\}$ as a generalized normalized orthogonal system in $\mathfrak{H}$, and when $\{\mathfrak{q}(U)\}$ is complete in $\mathfrak{H}$, then any vector $\mathfrak{f}$ in $\mathfrak{H}$ is expressed in the form

$$\mathfrak{f} = \int_V \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)}$$

where
$$\xi(U) = \big(\mathfrak{f}, \mathfrak{q}(U)\big). \tag{1.2}$$

Then $\xi(U)$ may be called the *representative* of $\mathfrak{f}$; it belongs to the space of differential set functions $\mathfrak{L}_2(\sigma)$ which is isomorph to $\mathfrak{H}$.[2] That is, if $\xi(U)$ and $\eta(U)$ are representatives of $\mathfrak{f}$ and $\mathfrak{g}$ respectively, then

$$(\mathfrak{f}, \mathfrak{g}) = (\xi, \eta) = \int_V \frac{\xi(dU)\overline{\eta(dU)}}{\sigma(dU)}. \tag{1.3}$$

2. Denote by $\mathfrak{M}'(\mathfrak{q})$ the linear manifold determined by the system $\{\mathfrak{q}(U)\}$, U being any set in the differential set system. Since $\{\mathfrak{q}(U)\}$ is complete in $\mathfrak{H}$, $\mathfrak{M}'(\mathfrak{q})$ is dense in $\mathfrak{H}$.

Let $\boldsymbol{T}$ be a linear operator which has its adjoint $\boldsymbol{T}^*$, and their domains $\mathfrak{D}$ and $\mathfrak{D}^*$ contain $\mathfrak{M}'(\mathfrak{q})$. Put

$$\big(\boldsymbol{T}\mathfrak{q}(U'), \mathfrak{q}(U)\big) = \mathfrak{K}(U, U'), \qquad \big(\boldsymbol{T}^*\mathfrak{q}(U'), \mathfrak{q}(U)\big) = \mathfrak{K}^*(U, U'). \tag{2.1}$$

Then by (1.2) $\mathfrak{K}(U, _{(U')})$[3] and $\mathfrak{K}^*(U, _{(U')})$ belong to $\mathfrak{L}_2(\sigma)$. Since

$$\mathfrak{K}(U, U') = \overline{\mathfrak{K}^*(U', U)},$$

(1) To prove this, consider two decompositions of V such as $V = U + \sum_n U_n$, $V = U' + \sum_m U'_m$. Then $U = UU' + \sum_m UU'_m$, $U' = UU' + \sum_n U_n U'$. Hence $\mathfrak{q}(U)\,[=]\,\mathfrak{q}(UU') + \sum_m \mathfrak{q}(UU'_m)$, $\mathfrak{q}(U')\,[=]\,\mathfrak{q}(UU') + \sum_n \mathfrak{q}(U_n U')$. Therefore, since $\big(\mathfrak{q}(U_i), \mathfrak{q}(U_j)\big) = 0$ when $U_i U_j = 0$, we have $\big(\mathfrak{q}(U), \mathfrak{q}(U')\big) = \big(\mathfrak{q}(UU'), \mathfrak{q}(UU')\big) = \sigma(UU')$. ([=] means the strong convergence of the series.)

(2) Cf. F. Maeda, this journal, **4** (1934), 69–75; this volume, 35.

(3) $\mathfrak{K}(U, _{(U')})$ means $\mathfrak{K}(U, U')$ considered as a function of set U, U' being a parameter. Similarly for $\mathfrak{K}(_{(U)}, U')$.

$\mathfrak{K}({}_{(U)}, U')$ belongs also to $\mathfrak{L}_2(\sigma)$. Thus $\mathfrak{K}(U, U')$ belongs to $\mathfrak{L}_2(\sigma)$, as a function of set U and as a function of set U'. In this case we say that $\mathfrak{K}(U, U')$ belongs to $\mathfrak{L}_2(\sigma, \sigma)$.

Let $\mathfrak{f}$ be any vector in $\mathfrak{D}$, then by (1.2)

$$\xi(U) = \big(\mathfrak{f}, \mathfrak{q}(U)\big), \qquad \eta(U) = \big(T\mathfrak{f}, \mathfrak{q}(U)\big)$$

are the representatives of $\mathfrak{f}$ and $T\mathfrak{f}$ respectively, and by (2.1) $\mathfrak{K}^*(U, {}_{(U')})$ is the representative of $T^*\mathfrak{q}(U')$. Hence by (1.3)

$$\eta(U') = \big(\mathfrak{f}, T^*\mathfrak{q}(U')\big) = \big(\xi(U), \mathfrak{K}^*(U, {}_{(U')})\big).$$

That is, $$\eta(U) = \int_V \frac{\mathfrak{K}(U, dU')\xi(dU')}{\sigma(dU')}. \tag{2.2}$$

Thus $\mathfrak{K}(U, U')$ is the kernel of the integral operator which transforms $\xi(U)$ to $\eta(U)$. $\mathfrak{K}(U, U')$ may be called the *representative* of T. Similarly $\mathfrak{K}^*(U, U')$ is the representative of T^*.(1) When T is self-adjoint, we have

$$\mathfrak{K}(U, U') = \overline{\mathfrak{K}(U', U)}.$$

3. Let T_1 and T_2 have their adjoints T_1^* and T_2^*, and their domains contain $\mathfrak{M}'(\mathfrak{q})$, and the representatives of T_1 and T_2 be $\mathfrak{K}_1(U, U')$ and $\mathfrak{K}_2(U, U')$ respectively. When T_1T_2 has its adjoint $(T_1T_2)^*$ and their domains contain $\mathfrak{M}'(\mathfrak{q})$, the representative $\mathfrak{K}(U, U')$ of T_1T_2 are defined by

$$\mathfrak{K}(U, U') = \big(T_1T_2\mathfrak{q}(U'), \mathfrak{q}(U)\big) = \big(T_2\mathfrak{q}(U'), T_1^*\mathfrak{q}(U)\big).$$

Since $\mathfrak{K}_2(U'', {}_{(U')})$ and $\mathfrak{K}_1^*(U'', {}_{(U)})$ are the representatives of $T_2\mathfrak{q}(U')$ and $T_1^*\mathfrak{q}(U)$ respectively, we have, by (1.3),

$$\mathfrak{K}(U, U') = \int_V \frac{\mathfrak{K}_1(U, dU'')\mathfrak{K}_2(dU'', U')}{\sigma(dU'')}. \tag{3.1}$$

This corresponds to the *matrix multiplication*. I will write (3.1) in

(1) When $\mathfrak{H}$ is $\mathfrak{L}_2(\sigma)$, we can take $\sigma(EU)$ as $\mathfrak{q}(U)$. Then

$$(T\mathfrak{q}(U'), \mathfrak{q}(U)) = (T\sigma(EU'), \sigma(EU)) = T\sigma(UU').$$

Hence the method (2.1) is the same as that of the preceding paper. (F. Maeda, this journal, **5** (1935), 112.)

an abbreviated form

$$\mathfrak{K}(U, U') = \mathfrak{K}_1\mathfrak{K}_2(U, U').$$

4. Let $\mathfrak{p}(E)$ be another completely additive vector valued differential set function defined in an abstract space Ω.[1] If $\{\mathfrak{p}(E)\}$ is complete in $\mathfrak{H}$, then any vector $\mathfrak{f}$ has for its representative

$$\psi(E) = \big(\mathfrak{f}, \mathfrak{p}(E)\big) \tag{4.1}$$

which belongs to $\mathfrak{L}_2(\beta)$, where $\beta(E) = \|\mathfrak{p}(E)\|^2$. Let us call it $\mathfrak{p}$-representation, and that of the preceding section $\mathfrak{q}$-representation.

Each $\mathfrak{q}(U)$ itself has a representative in the $\mathfrak{p}$-representation. We may write this $\mathfrak{p}$-representative $\mathfrak{U}(E, {}_{(U)})$. We shall then have by (4.1)

$$\mathfrak{U}(E, U) = \big(\mathfrak{q}(U), \mathfrak{p}(E)\big), \tag{4.2}$$

and $\mathfrak{U}(E, {}_{(U)})$ belongs to $\mathfrak{L}_2(\beta)$. Similarly, the $\mathfrak{q}$-representative of $\mathfrak{p}(E)$ is

$$\mathfrak{U}^*(U, E) = \big(\mathfrak{p}(E), \mathfrak{q}(U)\big), \tag{4.3}$$

and $\mathfrak{U}^*(U, {}_{(E)})$ belongs to $\mathfrak{L}_2(\sigma)$. From (4.2) and (4.3) we have

$$\mathfrak{U}(E, U) = \overline{\mathfrak{U}^*(U, E)}.$$

Since $\mathfrak{U}(E, {}_{(U)})$ and $\mathfrak{U}(E, {}_{(U')})$ are $\mathfrak{p}$-representatives of $\mathfrak{q}(U)$ and $\mathfrak{q}(U')$ respectively, we have

$$\big(\mathfrak{q}(U), \mathfrak{q}(U')\big) = \big(\mathfrak{U}(E, {}_{(U)}), \mathfrak{U}(E, {}_{(U')})\big).$$

Hence by (1.1) $$\big(\mathfrak{U}(E, {}_{(U)}), \mathfrak{U}(E, {}_{(U')})\big) = \sigma(UU'). \tag{4.4}$$

Similarly, from (4.3) we have

$$\big(\mathfrak{U}^*(U, {}_{(E)}), \mathfrak{U}^*(U, {}_{(E')})\big) = \beta(EE'). \tag{4.5}$$

Let $\xi(U)$ and $\psi(E)$ be $\mathfrak{q}$- and $\mathfrak{p}$-representative of a vector $\mathfrak{f}$ in $\mathfrak{H}$. Then, since $\mathfrak{U}^*(U, {}_{(E)})$ is the $\mathfrak{q}$-representative of $\mathfrak{p}(E)$, we have

$$\psi(E) = \big(\mathfrak{f}, \mathfrak{p}(E)\big) = \int_V \frac{\mathfrak{U}(E, dU)\xi(dU)}{\sigma(dU)}. \tag{4.6}$$

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(1) Ω may or may not be the same as the abstract space V.

Similarly, we have

$$\xi(U) = \int_{\Omega} \frac{\mathfrak{U}^*(U, dE)\psi(dE)}{\beta(dE)}. \tag{4.7}$$

The two representatives $\xi(U)$ and $\psi(E)$ are transformed by $\mathfrak{U}(E, U)$ and $\mathfrak{U}^*(U, E)$ from one to the other. Hence, as Dirac says,(1) we may call $\mathfrak{U}(E, U)$ and $\mathfrak{U}^*(U, E)$ *transformation functions.*

(4.4) shows that $\{\mathfrak{U}(E, {}_{(U)})\}$ is a generalized normalized orthogonal system in $\mathfrak{L}_2(\beta)$ with base $\sigma(U)$, and (4.5) shows that it is complete in $\mathfrak{L}_2(\beta)$; similarly for $\{\mathfrak{U}^*(U, {}_{(E)})\}$.(2) Or we may say that $\mathfrak{U}(E, U)$ and $\mathfrak{U}^*(U, E)$ are kernels of unitary transformations between $\mathfrak{L}_2(\sigma)$ and $\mathfrak{L}_2(\beta)$.(3)

5. When the domain of $\boldsymbol{T}$ contains $\mathfrak{M}'(\mathfrak{q})$ and the domain of its adjoint $\boldsymbol{T}^*$ contains $\mathfrak{M}'(\mathfrak{p})$, put

$$\big(\boldsymbol{T}\mathfrak{q}(U), \mathfrak{p}(E)\big) = \mathfrak{B}(E, U), \qquad \big(\boldsymbol{T}^*\mathfrak{p}(E), \mathfrak{q}(U)\big) = \mathfrak{B}^*(U, E), \tag{5.1}$$

then

$$\mathfrak{B}(E, U) = \overline{\mathfrak{B}^*(U, E)},$$

and $\mathfrak{B}(E, U)$ belongs to $\mathfrak{L}_2(\beta, \sigma)$. Let $\mathfrak{f}$ be any vector in the domain of $\boldsymbol{T}$, $\xi(U)$ being the $\mathfrak{q}$-representative of $\mathfrak{f}$. And let $\psi(E)$ be the $\mathfrak{p}$-representative of $\boldsymbol{T}\mathfrak{f}$. Then, since $\mathfrak{B}^*(U, {}_{(E)})$ is the $\mathfrak{q}$-representative of $\boldsymbol{T}^*\mathfrak{p}(E)$, we have

$$\psi(E) = \big(\boldsymbol{T}\mathfrak{f}, \mathfrak{p}(E)\big) = \big(\mathfrak{f}, \boldsymbol{T}^*\mathfrak{p}(E)\big) = \int_V \frac{\mathfrak{B}(E, dU)\xi(dU)}{\sigma(dU)}.$$

Hence $\mathfrak{B}(E, U)$ is the kernel of the integrating operator which transforms $\xi(U)$ to $\psi(E)$. We may call it, as Dirac does,(4) $\mathfrak{B}(E, U)$ a *mixed representative* of $\boldsymbol{T}$.(5)

Using the mixed representative of $\boldsymbol{T}$, we can find the relation between the $\mathfrak{q}$-representative $\mathfrak{K}(U, U')$ of $\boldsymbol{T}$ and the $\mathfrak{p}$-representative

(1) Dirac, loc. cit., 61 and 81.
(2) Cf. F. Maeda, this journal, **3** (1933), 253.
(3) Cf. F. Maeda, this journal, **5** (1935), 115–116.
(4) Dirac, loc. cit., 64.
(5) From (4.2) and (5.1), we see that the transformation function $\mathfrak{U}(E, U)$ is nothing but the mixed representative of the identical operator.

$\mathfrak{A}(E, E')$ of $\boldsymbol{T}$, where

$$\mathfrak{K}(U, U') = \big(\boldsymbol{T}\mathfrak{q}(U'), \mathfrak{q}(U)\big), \qquad \mathfrak{A}(E, E') = \big(\boldsymbol{T}\mathfrak{p}(E'), \mathfrak{p}(E)\big). \tag{5.2}$$

Since from (5.1) and (5.2), $\mathfrak{B}(E, {}_{(U')})$ and $\mathfrak{K}(U, {}_{(U')})$ are $\mathfrak{p}$- and $\mathfrak{q}$-representatives of $\boldsymbol{T}\mathfrak{q}(U')$ respectively, we have by (4.6)

$$\mathfrak{B}(E, U') = \int_V \frac{\mathfrak{U}(E, dU)\mathfrak{K}(dU, U')}{\sigma(dU)} = \mathfrak{U}\mathfrak{K}(E, U'). \tag{5.3}$$

Since from (5.2) and (5.1), $\mathfrak{A}^*(E, {}_{(E')})$ and $\mathfrak{B}^*(U, {}_{(E')})$ are $\mathfrak{p}$- and $\mathfrak{q}$-representatives of $\boldsymbol{T}^*\mathfrak{p}(E')$ respectively, we have by (4.6)

$$\mathfrak{A}^*(E, E') = \int_V \frac{\mathfrak{U}(E, dU)\mathfrak{B}^*(dU, E')}{\sigma(dU)}.$$

That is

$$\mathfrak{A}(E, E') = \int_V \frac{\mathfrak{B}(E, dU)\mathfrak{U}^*(dU, E')}{\sigma(dU)} = \mathfrak{B}\mathfrak{U}^*(E, E'). \tag{5.4}$$

(5.3) and (5.4) show the relations between $\mathfrak{q}$- and $\mathfrak{p}$-representatives and the mixed representative. Combining these two relations, we may write

$$\mathfrak{A}(E, E') = \mathfrak{U}\mathfrak{K}\mathfrak{U}^*(E, E').^{(1)}$$

Similarly, we have $\quad \mathfrak{K}(U, U') = \mathfrak{U}^*\mathfrak{A}\mathfrak{U}(U, U')$.

Representations of Linear Operators in the General Case.

6. The representation that we have used up to the present have all been the simple case; namely, the basic of the representation is

(1) Strictly speaking, indicating the order of integration, we must write $\mathfrak{A}(E, E') = (\mathfrak{U}\mathfrak{K})\mathfrak{U}^*(E, E')$. But the order of integration can be changed as follows:

Put $\qquad (\boldsymbol{T}\mathfrak{p}(E), \mathfrak{q}(U)) = \mathfrak{W}(U, E), \qquad (\boldsymbol{T}^*\mathfrak{q}(U), \mathfrak{p}(E)) = \mathfrak{W}^*(E, U).$

Then, as above, we have

$$\mathfrak{W}^*(E, U') = \int_V \frac{\mathfrak{U}(E, dU)\mathfrak{K}^*(dU, U')}{\sigma(dU)},$$

that is

$$\mathfrak{W}(U', E) = \int_V \frac{\mathfrak{K}(U', dU)\mathfrak{U}^*(dU, E)}{\sigma(dU)} = \mathfrak{K}\mathfrak{U}^*(U', E)$$

and

$$\mathfrak{A}(E, E') = \int_V \frac{\mathfrak{U}(E, dU)\mathfrak{W}(dU, E')}{\sigma(dU)} = \mathfrak{U}\mathfrak{W}(E, E').$$

Hence

$$\mathfrak{A}(E, E') = \mathfrak{U}(\mathfrak{K}\mathfrak{U}^*)(E, E')$$

composed with one completely additive vector valued differential set function $\mathfrak{q}(U)$. In the general case, let

$$\mathfrak{q}_1(U),\ \mathfrak{q}_2(U),\ \ldots\ldots,\ \mathfrak{q}_i(U),\ \ldots\ldots \tag{6.1}$$

be a finite or enumerably infinite sequence of completely additive vector valued differential set functions defined in a common differential set system in an abstract space V. Denote the closed linear manifold determined by the system $\{\mathfrak{q}_i(U)\}$, U being the parameter, by $\mathfrak{M}(\mathfrak{q}_i)$.

When
$$\mathfrak{M}(\mathfrak{q}_1),\ \mathfrak{M}(\mathfrak{q}_2),\ \ldots\ldots,\ \mathfrak{M}(\mathfrak{q}_i),\ \ldots\ldots \tag{6.2}$$

are mutually orthogonal, we say that (6.1) is an orthogonal system. If there exists no vector orthogonal to all the closed linear manifolds of (6.2), then we say that the orthogonal system (6.1) is complete in $\mathfrak{H}$.

In this case, we can take (6.1) as the basic of the representation. In my previous paper,[1] I have proved that

(i) for any vector $\mathfrak{f}$ in $\mathfrak{H}$,

$$\mathfrak{f}\ [=]\ \sum_i \int_V \frac{\xi_i(dU)\mathfrak{q}_i(dU)}{\sigma_i(dU)}\ ,^{(2)} \tag{6.3}$$

where $\xi_i(U) = \big(\mathfrak{f}, \mathfrak{q}_i(U)\big)$, $\sigma_i(U) = \|\mathfrak{q}_i(U)\|^2$; and $\xi_i(U)$ belongs to $\mathfrak{L}_2(\sigma_i)$;

(ii) for any pair $\mathfrak{f}, \mathfrak{g}$ in $\mathfrak{H}$, the identity

$$(\mathfrak{f}, \mathfrak{g}) = \sum_i (\xi_i, \eta_i) \tag{6.4}$$

is true, where

$$\xi_i(U) = \big(\mathfrak{f}, \mathfrak{q}_i(U)\big)\,, \qquad \eta_i(U) = \big(\mathfrak{g}, \mathfrak{q}_i(U)\big)\,.$$

Hence, from (6.3) the sequence of differential set functions $\{\xi_i(U)\}$ may be considered as the *representative* of $\mathfrak{f}$.

7. Denote by $\mathfrak{M}'(\mathfrak{q}_i)$ the linear manifold determined by the system $\{\mathfrak{q}_i(U)\}$, U being the parameter. Let $\boldsymbol{T}$ be a linear operator which has its adjoint $\boldsymbol{T}^*$, and their domains $\mathfrak{D}$ and $\mathfrak{D}^*$ contain all $\mathfrak{M}'(\mathfrak{q}_1)$, $\mathfrak{M}'(\mathfrak{q}_2)$, ..., $\mathfrak{M}'(\mathfrak{q}_i)$, ... Put

(1) F. Maeda, this journal, **4** (1934), 74.

(2) [=] means the strong convergence, when the right hand expression is an infinite series.

$$\big(T\mathfrak{q}_j(U'), \mathfrak{q}_i(U)\big) = \mathfrak{K}_{ij}(U, U'), \qquad \big(T^*\mathfrak{q}_j(U'), \mathfrak{q}_i(U)\big) = \mathfrak{K}^*_{ij}(U, U'). \tag{7.1}$$

Then $\mathfrak{K}_{ij}(U, {}_{(U')})$ and $\mathfrak{K}^*_{ij}(U, {}_{(U')})$ belong to $\mathfrak{L}_2(\sigma_i)$. Since

$$\mathfrak{K}_{ij}(U, U') = \overline{\mathfrak{K}^*_{ji}(U', U)},$$

$\mathfrak{K}_{ij}({}_{(U)}, U')$ belongs to $\mathfrak{L}_2(\sigma_j)$. Therefore $\mathfrak{K}_{ij}(U, U')$ belongs to $\mathfrak{L}_2(\sigma_i, \sigma_j)$.

Let $\mathfrak{f}$ be any vector in $\mathfrak{D}$ and let $\{\xi_i(U)\}$ and $\{\eta_i(U)\}$ be the representatives of $\mathfrak{f}$ and $T\mathfrak{f}$ respectively. Then since from (7.1) $\{\mathfrak{K}^*_{ij}(U, {}_{(U')})\}$ is the representative of $T^*\mathfrak{q}_j(U')$, from (6.4) we have

$$\eta_j(U) = \big(T\mathfrak{f}, \mathfrak{q}_j(U)\big) = \big(\mathfrak{f}, T^*\mathfrak{q}_j(U)\big)$$

$$= \sum_i \int_V \frac{\xi_i(dU')\overline{\mathfrak{K}^*_{ij}(dU', U)}}{\sigma_i(dU)} = \sum_i \int_V \frac{\mathfrak{K}_{ji}(U, dU')\xi_i(dU')}{\sigma_i(dU)}.$$

This can be written in the abbreviated form, i and j being interchanged,

$$\eta_i(U) = \sum_j \mathfrak{K}_{ij}\xi_j(U).$$

Hence we may call $\{\mathfrak{K}_{ij}(U, U')\}$ the *representative* of T. Similarly $\{\mathfrak{K}^*_{ij}(U, U')\}$ is the representative of T^*. When T is self-adjoint, they satisfy the relations

$$\mathfrak{K}_{ij}(U, U') = \overline{\mathfrak{K}_{ji}(U', U)} \qquad (i, j = 1, 2, \ldots\ldots).$$

When $\{\mathfrak{K}^{(1)}_{ij}(U, U')\}$ and $\{\mathfrak{K}^{(2)}_{ij}(U, U')\}$ are the representatives of $T^{(1)}$ and $T^{(2)}$, we can prove as sec. 3, the representative $\{\mathfrak{K}_{ij}(U, U')\}$ of $T^{(1)}T^{(2)}$ is expressed as follows:

$$\mathfrak{K}_{ij}(U, U') = \sum_k \mathfrak{K}^{(1)}_{ik}\mathfrak{K}^{(2)}_{kj}(U, U').$$

8. Let $\mathfrak{p}_1(E), \mathfrak{p}_2(E), \ldots\ldots, \mathfrak{p}_\nu(E), \ldots\ldots$ be another orthogonal system of vector valued differential set functions defined in an abstract space Ω, and let it be complete in $\mathfrak{H}$. Then the representative of $\mathfrak{f}$ is $\{\psi_\nu(E)\}$, where

$$\psi_\nu(E) = \big(\mathfrak{f}, \mathfrak{p}_\nu(E)\big) \qquad (\nu = 1, 2, \ldots\ldots),$$

and $\psi_\nu(E)$ belong to $\mathfrak{L}_2(\beta_\nu)$ respectively, $\beta_\nu(E)$ being $\|\mathfrak{p}_\nu(E)\|^2$. Let us call it $\mathfrak{p}$-representative, and that of the preceding section $\mathfrak{q}$-representative.

As in sec. 4, put

$$\mathfrak{U}_{\nu i}(E, U) = \big(\mathfrak{q}_i(U), \mathfrak{p}_\nu(E)\big)$$
$$(\nu, i = 1, 2, \dots). \qquad (8.1)$$
$$\mathfrak{U}^*_{i\nu}(U, E) = \big(\mathfrak{p}_\nu(E), \mathfrak{q}_i(U)\big)$$

Then $\mathfrak{U}_{\nu i}(E, U)$ and $\mathfrak{U}^*_{i\nu}(U, E)$ belong to $\mathfrak{L}_2(\beta_\nu, \sigma_i)$ and $\mathfrak{L}_2(\sigma_i, \beta_\nu)$ respectively, and

$$\mathfrak{U}_{\nu i}(E, U) = \overline{\mathfrak{U}^*_{i\nu}(U, E)}\,.$$

Since from (8.1) $\{\mathfrak{U}_{\nu i}(E, {}_{(U)})\}$ is the $\mathfrak{p}$-representative of $\mathfrak{q}_i(U)$, from (6.4) we have

$$\big(\mathfrak{q}_i(U), \mathfrak{q}_j(U')\big) = \sum_\nu \big(\mathfrak{U}_{\nu i}(E, {}_{(U)}), \mathfrak{U}_{\nu j}(E, {}_{(U')})\big)\,.$$

But, since
$$\big(\mathfrak{q}_i(U), \mathfrak{q}_j(U')\big) = 0 \qquad \text{when} \quad i \neq j$$
$$= \sigma_i(UU') \qquad \text{when} \quad i = j\,,$$
we have

$$\sum_\nu \big(\mathfrak{U}_{\nu i}(E, {}_{(U)}), \mathfrak{U}_{\nu j}(E, {}_{(U')})\big) = 0 \qquad \text{when} \quad i \neq j$$
$$= \sigma_i(UU') \qquad \text{when} \quad i = j\,. \qquad (8.2)$$

Similarly, we have

$$\sum_i \big(\mathfrak{U}^*_{i\nu}(U, {}_{(E)}), \mathfrak{U}^*_{i\mu}(U, {}_{(E')})\big) = 0 \qquad \text{when} \quad \nu \neq \mu$$
$$= \beta_\nu(EE') \qquad \text{when} \quad \nu = \mu\,.$$

Let $\{\xi_i(U)\}$ and $\{\psi_\nu(E)\}$ be $\mathfrak{q}$- and $\mathfrak{p}$-representative of a vector $\mathfrak{f}$ in $\mathfrak{H}$. Then, since from (8.1) $\{\mathfrak{U}^*_{i\nu}(U, {}_{(E)})\}$ is the $\mathfrak{q}$-representative of $\mathfrak{p}_\nu(E)$, we have

$$\psi_\nu(E) = \big(\mathfrak{f}, \mathfrak{p}_\nu(E)\big) = \sum_i \int_V \frac{\mathfrak{U}_{\nu i}(E, dU)\xi_i(dU)}{\sigma_i(dU)} = \sum_i \mathfrak{U}_{\nu i}\xi_i(E)\,.$$

Similarly, we have

$$\xi_i(U) = \sum_\nu \int_\Omega \frac{\mathfrak{U}^*_{i\nu}(U, dE)\psi_\nu(dE)}{\beta_\nu(dE)} = \sum_\nu \mathfrak{U}^*_{i\nu}\psi_\nu(U)\,.$$

Hence we may call $\{\mathfrak{U}_{\nu i}(E, U)\}$ and $\{\mathfrak{U}^*_{i\nu}(U, E)\}$ the *transformation functions.*

As in sec. 5, the mixed representative of $\boldsymbol{T}$ is $\{\mathfrak{B}_{\nu i}(E, U)\}$ where

$$\mathfrak{B}_{\nu i}(E, U) = \big(\boldsymbol{T}\mathfrak{q}_i(U), \mathfrak{p}_\nu(E)\big)$$

which belongs to $\mathfrak{L}_2(\beta_\nu, \sigma_i)$. When $\{\xi_i(U)\}$ is the q-representative of $\mathfrak{f}$, and $\{\psi_\nu(E)\}$ is the $\mathfrak{p}$-representative of $\boldsymbol{T}\mathfrak{f}$, then we have

$$\psi_\nu(E) = \sum_i \int_V \frac{\mathfrak{B}_{\nu i}(E, dU)\xi_i(dU)}{\sigma_i(dU)} = \sum_i \mathfrak{B}_{\nu i}\xi_i(E)\,.$$

Representations of Self-Adjoint Operators and Physical Interpretations.

9. From Dirac's theory of quantum mechanics,[(1)] the states of a particular dynamical system at one instant of time are represented by a vector in a vector space $\mathfrak{H}$ which satisfies the three axioms of sec. 1, and if the representative vector of a state is multiplied by any number, not zero, the resulting vector will represent the same state. And each observable is represented by a self-adjoint operator that can operate on the vectors of $\mathfrak{H}$. And a real function f of the observable which is represented by the self-adjoint operator $\boldsymbol{H}$, is represented by the self-adjoint operator $f(\boldsymbol{H})$.[(2)]

It is already known that to each self-adjoint operator $\boldsymbol{H}$ there exists a resolution of identity $\boldsymbol{E}(U)$, which is defined for all open intervals and points in the space of real numbers R_1 so that

$$\boldsymbol{H}\mathfrak{f} = \int_{R_1} \lambda \boldsymbol{E}(dU)\mathfrak{f} \qquad (9.1)$$

for all $\mathfrak{f}$ in the domain of $\boldsymbol{H}$, that is, for all $\mathfrak{f}$ such that $\int_{R_1} \lambda^2 \,\|\, \boldsymbol{E}(dU)\mathfrak{f} \,\|^2$ are finite.[(3)] In this case $f(\boldsymbol{H})$ is defined by

$$f(\boldsymbol{H})\mathfrak{f} = \int_{R_1} f(\lambda)\boldsymbol{E}(dU)\mathfrak{f}\,.$$

(1) Dirac, loc. cit., 20–30. Instead of the product of ψ-vector and ϕ-vector, I use the inner product of vectors. And for the sake of exact mathematical treatment, I add the condition that the vector space is complete.

(2) Dirac, loc. cit., 38–42.

(3) Cf. F. Maeda, this volume, 39.

When a completely additive vector valued differential set function $\mathfrak{q}(U)$, defined in a differential set system composed of finite open intervals and points in R_1, is generated by $E(U)$, that is

$$E(U)\mathfrak{q}(U') = \mathfrak{q}(UU'),$$

then from (9.1) we have

$$\boldsymbol{H}\mathfrak{q}(U) = \int_U \lambda \mathfrak{q}(dU). \tag{9.2}$$

We may call $\mathfrak{q}(U)$ the eigenvector (characteristic vector) of $\boldsymbol{H}$.(1) When the set U is composed of one point λ_0, and $\mathfrak{q}(\lambda_0)$ is not a null vector, (9.2) becomes

$$\boldsymbol{H}\mathfrak{q}(\lambda_0) = \lambda_0 \mathfrak{q}(\lambda_0),$$

therefore $\mathfrak{q}(\lambda_0)$ is the eigenvector of $\boldsymbol{H}$ with respect to the eigenvalue λ_0.

The general assumption for the physical interpretation is the following :(2)

(A) If the measurement of the observable represented by $\boldsymbol{H}$, for the system in the state represented by the vector $\mathfrak{f}$, is made a large number of times, the average of all the results obtained will be $(\boldsymbol{H}\mathfrak{f}, \mathfrak{f})$, provided $\mathfrak{f}$ is normalized.(3)

10. If there exists an eigenvector $\mathfrak{q}(U)$ of $\boldsymbol{H}$ such that $\{\mathfrak{q}(U)\}$ is complete in $\mathfrak{H}$, then we say that $\boldsymbol{H}$ has a simple spectrum.(4) In this case, we may use $\mathfrak{q}(U)$ as the basic of the representation. And the representative $\mathfrak{R}(U, U')$ of $\boldsymbol{H}$ becomes, by (9.2)

$$\mathfrak{R}(U, U') = \big(\boldsymbol{H}\mathfrak{q}(U'), \mathfrak{q}(U)\big) = \int_{UU'} \lambda\sigma(dU), \tag{10.1}$$

which we may call the *diagonal representation.*(5)

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(1) Cf. ibid.
(2) Cf. Dirac, loc. cit., 43.
(3) Of course, $\mathfrak{f}$ must be in the domain of $\boldsymbol{H}$.
(4) Cf. M. H. Stone, *Linear Transformations in Hilbert Space*, (1932), 275.
(5) For, when $\boldsymbol{H}$ has only a discrete spectrum, whose eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$, if we take $\|\mathfrak{q}(\lambda_i)\|^2 = \sigma(\lambda_i) = 1$ for all i, then (10.1) becomes $\mathfrak{R}(\lambda_i, \lambda_j) = \lambda_i \delta_{ij}$.

Let $\mathfrak{f}$ be any vector in $\mathfrak{H}$. Then, from sec. 1 the representative of $\mathfrak{f}$ is

$$\xi(U) = \big(\mathfrak{f}, \mathfrak{q}(U)\big).$$

The physical interpretation of $\xi(U)$ may be stated as follows:

(P) *The probability of* $\boldsymbol{H}$ *having a value lying within a specified range* U *for the state* $\mathfrak{f}$, *is* $\displaystyle\int_U \frac{|\xi(dU)|^2}{\sigma(dU)}$, *provided* $\mathfrak{f}$ *is normalized.*

To prove this theorem, define $f(\lambda)$ as follows:

$$f(\lambda) = 1 \qquad \text{when } \lambda \text{ is in } U$$
$$= 0 \qquad \text{when } \lambda \text{ is not in } U.$$

Let $P(U)$ be the probability of $\boldsymbol{H}$ having a value within U for the state $\mathfrak{f}$. Then from the theory of probability $\int_{R_1} f(\lambda)P(dU) = P(U)$ is the average value of $f(\boldsymbol{H})$ for the state $\mathfrak{f}$.(1) But from the general assumption (A) the average value of $f(\boldsymbol{H})$ for the state $\mathfrak{f}$ is $\big(f(\boldsymbol{H})\mathfrak{f}, \mathfrak{f}\big)$.(2)

Since
$$\mathfrak{f} = \int_{R_1} \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)}, \tag{10.2}$$

we have
$$\boldsymbol{E}(U)\mathfrak{f} = \int_U \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)}.^{(3)} \tag{10.3}$$

And
$$f(\boldsymbol{H})\mathfrak{f} = \int_{R_1} f(\lambda)\boldsymbol{E}(dU)\mathfrak{f} = \boldsymbol{E}(U)\mathfrak{f}.$$

Therefore from (10.2) and (10.3)

$$\big(f(\boldsymbol{H})\mathfrak{f}, \mathfrak{f}\big) = \big(\boldsymbol{E}(U)\mathfrak{f}, \mathfrak{f}\big) = \int_U \frac{|\xi(dU)|^2}{\sigma(dU)}.^{(4)}$$

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(1) Cf. J. v. Neumann, *Mathematische Grundlagen der Quantenmechanik*, (1932), 106–107.

(2) Since $f(\boldsymbol{H})$ is a projection, $\mathfrak{f}$ is in the domain of $f(\boldsymbol{H})$.

(3) Cf. F. Maeda, this volume, 38–39.

(4) By (1.3).

Thus we have the required result.[1]

Of course, $\int_U \frac{|\xi(dU)|^2}{\sigma(dU)}$ has the property of probability. First, the addition theorem of probability holds. For when $U=\sum U_i$,

$$\int_U \frac{|\xi(dU)|^2}{\sigma(dU)} = \sum \int_{U_i} \frac{|\xi(dU)|^2}{\sigma(dU)}.$$

And secondly, the probability of $\boldsymbol{H}$ having a value in R_1 is 1, for

$$\int_{R_1} \frac{|\xi(dU)|^2}{\sigma(dU)} = \|\xi\|^2 = \|\mathfrak{f}\|^2 = 1.$$

Thus the representative gives the probability in the above sense. Hence we may call the representative a *probability amplitude.*[2]

Take the state $\frac{\mathfrak{q}(U')}{\sqrt{\sigma(U')}}$, which is normalized, then the representative of it is $\left(\frac{\mathfrak{q}(U')}{\sqrt{\sigma(U')}}, \mathfrak{q}(U)\right) = \frac{\sigma(UU')}{\sqrt{\sigma(U')}}$. Hence the probability of $\boldsymbol{H}$ having a value lying within U for this state is

$$\frac{1}{\sigma(U')}\int_U \frac{|\sigma(dU\cdot U')|^2}{\sigma(dU)} = \frac{1}{\sigma(U')}\int_{UU'} \sigma(dU) = \frac{\sigma(UU')}{\sigma(U')}. \qquad (10.4)$$

Therefore, when $U \supseteqq U'$ the probability is 1.

11. Let $\boldsymbol{K}$ be another self-adjoint operator which has a simple spectrum, and let $\mathfrak{p}(E)$ be an eigenvector of $\boldsymbol{K}$ so that $\{\mathfrak{p}(E)\}$ is complete in $\mathfrak{H}$.[3] Then, by sec. 4 we have the transformation function

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(1) Conversely, we can deduce the general assumption (**A**) from the theorem (**P**) as follows: From (**P**) the average value of $\boldsymbol{H}$ for the state $\mathfrak{f}$ is $\int_{R_1} \frac{\lambda|\xi(dU)|^2}{\sigma(dU)}$. But, since the representative of $\mathfrak{f}$ is $\xi(U)$, and from (10.1) the representative of $\boldsymbol{H}$ is $\mathfrak{K}(U, U') = \int_{UU'} \lambda\sigma(dU)$, from (2.2) the representative of $\boldsymbol{H}\mathfrak{f}$ is

$$\eta(U) = \int_{R_1} \frac{\mathfrak{K}(U, dU')\xi(dU')}{\sigma(dU')} = \int_U \lambda\xi(dU).$$

Hence $(\boldsymbol{H}\mathfrak{f}, \mathfrak{f}) = (\eta, \xi) = \int_{R_1} \frac{\lambda|\xi(dU)|^2}{\sigma(dU)}$. Thus, we have (**A**).

(2) Cf. Dirac, loc. cit., 66.

(3) E is also a finite open interval or a point in R_1.

$$\mathfrak{U}(E, U) = \left(\mathfrak{q}(U), \mathfrak{p}(E)\right),$$

which satisfies the following relations

$$\left(\mathfrak{U}(E, _{(U)}), \mathfrak{U}(E, _{(U')})\right) = \sigma(UU'), \qquad \left(\mathfrak{U}(_{(E)}, U), \mathfrak{U}(_{(E')}, U)\right) = \beta(EE'). \tag{11.1}$$

The physical interpretation of the transformation function as a probability amplitude will be given easily from the preceding section. Since $\frac{1}{\sqrt{\sigma(U)}}\mathfrak{U}(E, _{(U)})$ is the $\mathfrak{p}$-representative of $\frac{\mathfrak{q}(U)}{\sqrt{\sigma(U)}}$, $\frac{1}{\sigma(U)}\int_E \frac{|\mathfrak{U}(dE, U)|^2}{\beta(dE)}$ is the probability of $\boldsymbol{K}$ having a value within the range E for the state $\frac{\mathfrak{q}(U)}{\sqrt{\sigma(U)}}$. But from (10.4) the probability of $\boldsymbol{H}$ having a value within U for the state $\frac{\mathfrak{q}(U)}{\sqrt{\sigma(U)}}$ is 1. Hence $\frac{1}{\sigma(U)}\int_E \frac{|\mathfrak{U}(dE, U)|^2}{\beta(dE)}$ *is the probability of* $\boldsymbol{K}$ *having a value within* E *for the state* $\frac{\mathfrak{q}(U)}{\sqrt{\sigma(U)}}$ *for which* $\boldsymbol{H}$ *certainly has a value within the range* U.(1)

Similarly for $\frac{1}{\beta(E)}\int_U \frac{|\mathfrak{U}^*(dU, E)|^2}{\sigma(dU)}$. Of course, $\frac{1}{\sigma(U)}\int_E \frac{|\mathfrak{U}(dE, U)|^2}{\beta(dE)}$ has the property of probability. It is evident that it satisfies the addition theorem. When $E = R_1$, since from (11.1)

$$\int_{R_1} \frac{|\mathfrak{U}(dE, U)|^2}{\beta(dE)} = \sigma(U),$$

the probability is 1.

12. When the self-adjoint operator has not the simple spectrum, and $\{\mathfrak{q}_1(U), \mathfrak{q}_2(U), \ldots, \mathfrak{q}_i(U), \ldots\}$ are the orthogonal system of eigenvectors of $\boldsymbol{H}$ which is complete in $\mathfrak{H}$, then from sec. 6, the representative of any vector $\mathfrak{f}$ is $\{\xi_i(U)\}$ where

$$\xi_i(U) = \left(\mathfrak{f}, \mathfrak{q}_i(U)\right) \qquad (i = 1, 2, \ldots).$$

(1) Cf. Dirac, loc. cit., 82.

The physical interpretation of this representative can be obtained as sec. 10, as follows:

The probability of $\boldsymbol{H}$ *having a value lying within a specified range* U *for the state* $\mathfrak{f}$, *is* $\sum_i \int_U \frac{|\xi_i(dU)|^2}{\sigma_i(dU)}$, *provided* $\mathfrak{f}$ *is normalized.*(1)

Let $\{\mathfrak{p}_1(E), \mathfrak{p}_2(E), \ldots, \mathfrak{p}_i(E), \ldots\}$ be the orthogonal system of eigenvectors of another self-adjoint operator $\boldsymbol{K}$, which is complete in $\mathfrak{H}$. Then from sec. 8 we have the transformation functions $\mathfrak{U}_{\nu i}(E, U)$, where

$$\mathfrak{U}_{\nu i}(E, U) = \big(\mathfrak{q}_i(U), \mathfrak{p}_\nu(E)\big) \qquad (\nu, i = 1, 2, \ldots).$$

Then, as sec. 11, $\frac{1}{\sigma_i(U)} \sum_\nu \int_E \frac{|\mathfrak{U}_{\nu i}(dE, U)|^2}{\beta_\nu(dE)}$ *is the probability*(2) *of* $\boldsymbol{K}$ *having a value within the range* E *for the state* $\frac{\mathfrak{q}_i(U)}{\sqrt{\sigma_i(U)}}$ *for which* $\boldsymbol{H}$ *certainly has a value within the range* U.(3) Similarly, for $\frac{1}{\beta_\nu(E)} \sum_i \int_U \frac{|\mathfrak{U}^*_{i\nu}(dU, E)|^2}{\sigma_i(dU)}$.

13. When the self-adjoint operators have not the simple spectrum, we also use the complete set of commuting self-adjoint operators.(4) Let $\boldsymbol{H}_1, \boldsymbol{H}_2, \ldots, \boldsymbol{H}_n$ be commuting operators, and $\boldsymbol{E}_1(U_1), \boldsymbol{E}_2(U_2), \ldots, \boldsymbol{E}_n(U_n)$ be the resolutions of identity defined in a differential set system in R_1, with respect to $\boldsymbol{H}_1, \boldsymbol{H}_2, \ldots, \boldsymbol{H}_n$. Then $(U_1, U_2, \ldots, U_n)$ is a set in the product space $(R_1, R_1, \ldots, R_1)$, that is the euclidean space of n dimensions, which we denote by R_n. Now we consider only the sets in R_n which have the form $(U_1, U_2, \ldots, U_n)$, and we denote it by U. We restrict the decomposition of $U = (U_1, U_2, \ldots, U_n)$ so that

$$(U_1, U_2, \ldots, U_n) = \sum_{i_1, i_2, \ldots, i_n} (U_1^{(i_1)}, U_2^{(i_2)}, \ldots, U_n^{(i_n)})$$

(1) When $U = R_1$, from (6.4) the probability is 1.

(2) When $E = R_1$, from (8.2) the probability is 1.

(3) Denote the $\mathfrak{q}$-representative of $\frac{\mathfrak{q}_i(U')}{\sqrt{\sigma_i(U')}}$ by $\{\xi_j(U)\}$, then $\xi_j(U) = 0$ when $j \neq i$, $\xi_i(U) = \frac{\sigma_i(UU')}{\sqrt{\sigma_i(U')}}$. Hence the probability of $\boldsymbol{H}$ having a value in U' for the state $\frac{\mathfrak{q}_i(U')}{\sqrt{\sigma_i(U')}}$ is $\sum_j \int_{U'} \frac{|\xi_j(dU)|^2}{\sigma_j(dU)} = \frac{1}{\sigma_i(U')} \int_{U'} \frac{|\sigma_i(U' \cdot dU)|^2}{\sigma(dU)} = 1$.

(4) Cf. Dirac, loc. cit., 58.

where $U_1 = \sum_{i_1} U_1^{(i_1)},\ U_2 = \sum_{i_2} U_2^{(i_2)},\ \ldots\ldots,\ U_n = \sum_{i_n} U_n^{(i_n)}$.

Since $E_1(U_1)E_2(U_2)\ldots\ldots E_n(U_n)$ is a projection defined for $U = (U_1, U_2, \ldots\ldots, U_n)$,(1) we denote it by $E(U)$. Then we can easily prove that $E(U)$ is a resolution of identity defined in R_n.

If there exists a completely additive vector valued set function $\mathfrak{q}(U)$ in R_n generated by $E(U)$, so that $\{\mathfrak{q}(U)\}$ is complete in $\mathfrak{H}$, then we say that $H_1, H_2, \ldots\ldots, H_n$ is a *complete set of commuting self-adjoint operators.*

In this case $\mathfrak{q}(U) = \mathfrak{q}(U_1, U_2, \ldots\ldots, U_n)$ is a common eigenvector to all $H_1, H_2, \ldots\ldots, H_n$. For, since

$$E_\nu(U_\nu)\mathfrak{q}(U_1', \ldots\ldots, U_\nu', \ldots\ldots, U_n') = \mathfrak{q}(U_1', \ldots\ldots, U_\nu U_\nu', \ldots\ldots, U_n'),$$

from
$$H_\nu \mathfrak{f} = \int_{R_1} \lambda_\nu E_\nu(dU_\nu)\mathfrak{f},$$

we have

$$H_\nu \mathfrak{q}(U_1, \ldots\ldots, U_\nu, \ldots\ldots, U_n) = \int_{U_\nu} \lambda_\nu \mathfrak{q}(U_1, \ldots\ldots, dU_\nu, \ldots\ldots, U_n) \quad (13.1)$$

for $\nu = 1, 2, \ldots\ldots, n$.

14. Since the representation theory of sections 1–5 is the general theory with respect to $\mathfrak{q}(U)$ defined in an abstract space V, this theory can be applied to the case where $\mathfrak{q}(U)$ is defined in R_n as in the preceding section.(2)

Thus the representative $\mathfrak{R}_\nu(U, U') = \mathfrak{R}_\nu(U_1, U_2, \ldots\ldots, U_n;\ U_1', U_2', \ldots\ldots, U_n')$ of H_ν is

$$\mathfrak{R}_\nu(U, U') = \big(H_\nu \mathfrak{q}(U'), \mathfrak{q}(U)\big) = \int_{U_\nu U_\nu'} \lambda_\nu \sigma(U_1U_1', \ldots\ldots, dU_\nu, \ldots\ldots, U_nU_n')^{(3)}$$

from (13.1). This may be called the *diagonal representative.*

From sec. 2, the $\mathfrak{q}$-representative of any vector $\mathfrak{f}$ is

$$\xi(U) = \xi(U_1, U_2, \ldots\ldots, U_n) = \big(\mathfrak{f}, \mathfrak{q}(U)\big).$$

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(1) Cf. Stone, loc. cit., 71, 301.

(2) In this case, the decompositions used for the definition of integrals must be restricted to the forms cited in the preceding section.

(3) This integral can be obtained easily from the definition of the integral.

And as sec. 10, $\int_U \frac{|\mathfrak{x}(dU)|^2}{\sigma(dU)} = \int_{(U_1, U_2, \ldots, U_n)} \frac{|\mathfrak{x}(dU_1, dU_2, \ldots, dU_n)|^2}{\sigma(dU_1, dU_2, \ldots, dU_n)}$ *is the probability of a simultaneous observation of* $\boldsymbol{H}_1, \boldsymbol{H}_2, \ldots, \boldsymbol{H}_n$ *for the state* $\mathfrak{f}$ *yielding for each* $\boldsymbol{H}_\nu$ *a result in the range* U_ν, *provide* $\mathfrak{f}$ *is normalized.*(1)

Let $\boldsymbol{K}_1, \boldsymbol{K}_2, \ldots, \boldsymbol{K}_m$ be another complete set of commuting observables, and let $\mathfrak{p}(E) = \mathfrak{p}(E_1, E_2, \ldots, E_m)$ be a common eigenvector of $\boldsymbol{K}_1, \boldsymbol{K}_2, \ldots, \boldsymbol{K}_m$ so that $\{\mathfrak{p}(E)\}$ is complete in $\mathfrak{H}$. Then, as sec. 4 we have a transformation function

$$\mathfrak{U}(E, U) = \mathfrak{U}(E_1, E_2, \ldots, E_m;\ U_1, U_2, \ldots, U_n) = \big(\mathfrak{q}(U), \mathfrak{p}(E)\big).$$

$\mathfrak{U}(E, U)$ has the following physical interpretation.

$$\frac{1}{\sigma(U_1, U_2, \ldots, U_n)} \int_{(E_1, E_2, \ldots, E_m)} \frac{|\mathfrak{U}(dE_1, dE_2, \ldots, dE_m;\ U_1, U_2, \ldots, U_n)|^2}{\beta(dE_1, dE_2, \ldots, dE_m)}$$

is the probability of $\boldsymbol{K}_1, \boldsymbol{K}_2, \ldots, \boldsymbol{K}_m$ *having the values within the range* $E_1, E_2, \ldots, E_m$ *respectively for the state* $\frac{\mathfrak{q}(U_1, U_2, \ldots, U_n)}{\sqrt{\sigma(U_1, U_2, \ldots, U_n)}}$ *for which* $\boldsymbol{H}_1, \boldsymbol{H}_2, \ldots, \boldsymbol{H}_n$ *certainly have values within* $U_1, U_2, \ldots, U_n$ *respectively.*(2)

15. Lastly, as an application of the physical interpretation of the representatives as the probability amplitude, we deduce one form of the principle of uncertainty.

For the sake of simplicity, let us take a dynamical system which has one degree of freedom. Then it is described by a coordinate q and its conjugate momentum p. Then we have two self-adjoint operators $\boldsymbol{Q}$ and $\boldsymbol{P}$ which are defined by

$$\boldsymbol{Q}f(q) = qf(q), \qquad \boldsymbol{P}f(q) = \frac{h}{2\pi i}\frac{d}{dq}f(q),$$

where $f(q)$ is the wave function representing the state. Let $\beta(E)$ be

(1) Cf. Dirac, loc. cit., 65 and 83.

(2) Cf. Dirac, loc. cit., 66.

the length of the interval E, and take the differential set function $\psi(E)$ in $\mathfrak{L}_2(\beta)$ so that

$$\psi(E) = \int_E f(q)dq\,.$$

Then as we have seen in my previous paper,(1) in the space of the differential set functions $\mathfrak{L}_2(\beta)$, $\beta(EE')$ is the eigenfunction of $\boldsymbol{Q}$, and the kernel of $\boldsymbol{Q}$ is

$$\mathfrak{Q}(E, E') = \int_{EE'} q\beta(dE)\,.$$

Since this is nothing else than the diagonal representative of $\boldsymbol{Q}$, $\psi(E)$ is the representative of the state with respect to the eigenvectors of $\boldsymbol{Q}$.

I have also shown in the previous paper,(2) that

$$\mathfrak{U}(E, U) = \frac{1}{\sqrt{h}}\int_U d\lambda \int_E e^{\frac{2\pi i}{h}\lambda q} dq \qquad (15.1)$$

is the eigenfunction of $\boldsymbol{P}$ which is complete in $\mathfrak{L}_2(\beta)$, so that

$$\boldsymbol{P}\mathfrak{U}(E, U) = \int_U \lambda \mathfrak{U}(E, dU) \qquad \text{and} \qquad \big(\mathfrak{U}(E, {}_{(U)}), \mathfrak{U}(E, {}_{(U')})\big) = \beta(UU')\,.$$

Then the transformation function connecting the representation in which $\boldsymbol{Q}$ is diagonal with that in which $\boldsymbol{P}$ is diagonal is

$$\big(\mathfrak{U}(E, U),\ \beta(EE')\big) = \mathfrak{U}(E', U)\,.^{(3)}$$

Let us now apply the physical interpretation of the transformation function $\mathfrak{U}(E', U)$ as the probability amplitude. From sec. 11, $\dfrac{1}{\beta(U)}\displaystyle\int_{E'} \dfrac{|\,\mathfrak{U}(dE', U)\,|^2}{\beta(dE')}$ is the probability of $\boldsymbol{Q}$ having a value within the range E' for the state $\dfrac{\mathfrak{U}(E, U)}{\sqrt{\beta(U)}}$ for which $\boldsymbol{P}$ certainly has a value within the range U. But from (15.1) we have

(1) F. Maeda, this volume, 42.
(2) Ibid., 44.
(3) Cf. (4.2), where $\mathfrak{q}(U) = \mathfrak{U}(E, U)$, $\mathfrak{p}(E') = \beta(EE')$.

$$\frac{1}{\beta(U)}\int_{E'}\frac{|\mathfrak{U}(dE', U)|^2}{\beta(dE')} = \frac{1}{\beta(U)}\int_{E'}\frac{1}{h}\left|\int_U e^{\frac{2\pi i}{h}\lambda q}d\lambda\right|^2 dq \leqq \frac{1}{h}\beta(E')\beta(U).$$

When we interchange $\boldsymbol{P}$ and $\boldsymbol{Q}$, and take the state $\frac{\mathfrak{U}^*(U, E')}{\sqrt{\beta(E')}}$, we have the same result.

But the above result concerns especially the states of the form $\frac{\mathfrak{U}(E, U)}{\sqrt{\beta(U)}}$ or $\frac{\mathfrak{U}^*(U, E')}{\sqrt{\beta(E')}}$. We have the same result for any state as follows.

Let $\psi(E)$ represent any state for which $\boldsymbol{Q}$ certainly has a value within the range E'. Then, $\psi(E)$ being normalized, from sec. 10 we have

$$\int_{E'}\frac{|\psi(dE)|^2}{\beta(dE)} = 1 = \int_{R_1}\frac{|\psi(dE)|^2}{\beta(dE)}.$$

Hence, $\psi(E)$ must vanish for any interval E outside E'. Now by (4.7) the representative $\xi(U)$ of $\psi(E)$ with respect to the eigenvectors of $\boldsymbol{P}$ is

$$\xi(U) = \int_{R_1}\frac{\mathfrak{U}^*(U, dE)\psi(dE)}{\beta(dE)} = \int_{E'}\frac{\mathfrak{U}^*(U, dE)\psi(dE)}{\beta(dE)}.$$

Let $\psi(E) = \int_E f(q)dq$, then by (15.1) we have

$$\xi(U) = \frac{1}{\sqrt{h}}\int_{E'} f(q)dq\int_U e^{-\frac{2\pi i}{h}\lambda q}d\lambda.$$

Then, since $\int_{E'}|f(q)|^2 dq = \|\psi\|^2 = 1$, we have

$$\int_U\frac{|\xi(dU)|^2}{\beta(dU)} = \frac{1}{h}\int_U\left|\int_{E'} f(q)e^{-\frac{2\pi i}{h}\lambda q}dq\right|^2 d\lambda \leqq \frac{1}{h}\beta(E')\beta(U).$$

Hence, taking $\Delta q = \beta(E')$, $\Delta p = \beta(U)$, from sec. 10, we have the following result.

The probability of $\boldsymbol{P}$ *having a value within the range of width* Δp *for any state for which* $\boldsymbol{Q}$ *certainly has a value within the range of width* Δq, *is not greater than* $\frac{\Delta q\Delta p}{h}$.

If we take instead of $\psi(E)$ the representative of the state with respect to the eigenvectors of $\boldsymbol{P}$, then we have the same result in which $\boldsymbol{P}$ and $\boldsymbol{Q}$ are interchanged. Therefore, the more accurately the value of $\boldsymbol{Q}$ is known, the greater the uncertainty of $\boldsymbol{P}$, and vice versa. Hence it seems to me, the above result is one form of the principle of uncertainty.(1)

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(1) If we take the state $\psi(E)$, so that

$$f(q) = \frac{1}{\sqrt{\beta(E')}} e^{\frac{2\pi i}{h} kq} \qquad \text{when } q \text{ is in } E'$$
$$= 0 \qquad \text{when } q \text{ is not in } E',$$

k being a constant, then we can find the lower bound of the probability $x = \int_U \frac{|\xi(dU)|^2}{\beta(dU)}$ as follows.

$$x = \frac{1}{h\beta(E')} \int_U \left| \int_{E'} e^{\frac{2\pi i}{h}(k-\lambda)q} dq \right|^2 d\lambda = \frac{h}{2\pi^2 \Delta q} \int_U \frac{1-\cos\frac{2\pi}{h}(k-\lambda)\Delta q}{(k-\lambda)^2} d\lambda,$$

where $\Delta q = \beta(E')$. Hence if we take $U = \left(k-\frac{1}{2}\Delta p,\ k+\frac{1}{2}\Delta p\right)$, then

$$x = \frac{1}{\pi} \int_{-\pi a}^{\pi a} \frac{1-\cos t}{t^2} dt$$

where $a = \frac{\Delta q \Delta p}{h}$. Since $\int_{-\infty}^{+\infty} \frac{1-\cos t}{t^2} dt = \pi$,

$$x = 1 - \frac{2}{\pi} \int_{\pi a}^{\infty} \frac{1-\cos t}{t^2} dt > 1 - \frac{4}{\pi^2 a}.$$