Wave Geometry.[1]
(Geometry in Microscopic Space.)

By

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§ 1. Introduction.

According to Prof. Mimura,[2] we define the expression for the metric in the general microscopic space by

$$ds\psi = \gamma_i dx^i \psi \tag{1.1}$$

where γ's are 4–4 matrices satisfying the equation

$$\gamma_{(i}\gamma_{j)} = g_{ij} I$$

and ψ is a 1–4 matrix given as a solution of the "*unknown Dirac equation*"—this equation will be obtained in § 5.

According to W. Pauli,[3] the most general expression for γ_i can be expressed as

$$\gamma_i = U h_i^{\varepsilon} \overset{\circ}{\gamma}_{\varepsilon} U^{-1} \tag{1.2}$$

where U is any 4–4 matrix, h_i^{ε} can be any functions and $\overset{\circ}{\gamma}_{\lambda}$ the Dirac matrices.[4] Hereafter we assume that Latin suffices vary 1–4 and Greek suffices 1–5.

For the sake of brevity in the subsequent calculation and also bearing in mind certain future applications to projective relativity, we introduce γ_5 such that

(1), (2) This Journal **5** (1935), 99.

(3) Ann. d. Phys. **18** (1933), 337.

(4) $\overset{\circ}{\gamma}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, $\overset{\circ}{\gamma}_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$, $\overset{\circ}{\gamma}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$,

$\overset{\circ}{\gamma}_4 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$, $\overset{\circ}{\gamma}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$,

$$\gamma_5 = Uh_5^{\varepsilon}\overset{\circ}{\gamma}_{\varepsilon}U^{-1}$$

where the rank of (h^{λ}_{μ}) is 5.

Thus we shall consider five matrices given by

$$\gamma_\lambda = Uh_\lambda^{\varepsilon}\overset{\circ}{\gamma}_{\varepsilon}U\,,$$

$$\gamma_{(\nu}\gamma_{\mu)} = g_{\nu\mu}I\,;$$

here the rank of $g_{\nu\mu}$ is 5.

§ 2. Coordinate transformations and ψ-transformations.

In *wave geometry* we assume that the transformation occurs in the form

$$\left.\begin{aligned} 'x^i &= f_i(x^1, \ldots\ldots, x^4) \\ '\psi &= \Phi(x, \psi) \end{aligned}\right\}. \tag{2.1}$$

Here we state the following assumption.

The transformation (2.1) *makes* $ds\psi = 0$ *always invariant.*

This can be expressed analytically

$$'(\gamma_i dx^i \psi) = \Omega\gamma_i dx^i\psi\,, \tag{2.2}$$

where Ω is in general a certain 4–4 matrix.

But since ψ may be functions of x^j and $'\psi$ is a 1–4 matrix, we can find a 4–4 matrix S composed of functions of x^j so that

$$'\psi = S\psi$$

Therefore (2.2) becomes

$$'\gamma_j P^j_i dx^i S\psi = \Omega\gamma_i dx^i\psi$$

hence

$$'\gamma_j = \overline{P}^i_j\Omega\gamma_i S^{-1} \tag{2.3}$$

where $P^j_i = \dfrac{\partial' x^j}{\partial x^i}$ and $\overline{P}^i_j = \dfrac{\partial x^i}{\partial' x^j}$, but on the other hand it must be that

$$'\gamma_{(i}{}'\gamma_{j)} = {}'g_{ij}I\,,$$

therefore, substituting (2.3) into the latter we have

$$\gamma_{(i} W \gamma_{j)} = W^{-1} g_{ij} , \tag{2.4}$$

where

$$W = S^{-1}\Omega , \tag{2.5}$$

or

$$W\gamma_i W = \gamma_i \tag{2.6}$$

[1]

If we can find W from the above, we can obtain Ω from (2.5) by giving S arbitrarily.

To solve (2.6) we assume for the time being that the transformation (2.1) is an infinitesimal transformation; then

$$W = I + \omega .$$

Substituting this into (2.6)

$$\omega\gamma_i + \gamma_i\omega = 0. \tag{2.7}$$

Expanding ω into sedenion

$$\omega = t^{\lambda\mu}\gamma_\lambda\gamma_\mu + t^\lambda\gamma_\lambda + tI$$

where

$$t^{\lambda\mu} = -t^{\mu\lambda} .$$

Substituting this into (2.7), and after some calculation, we have

$$\gamma_i t^{\nu\mu}\gamma_\nu\gamma_\mu + 2t^{\nu\mu}\gamma_\nu g_{\mu i} + t^\nu g_{\nu i} + t\gamma_i = 0 . \tag{2.8}$$

Multiplying γ_i and contracting for i, we have

$$4t^{\nu\mu}\gamma_\nu\gamma_\mu - 2t^{i\nu}\gamma_i\gamma_\nu + t^i\gamma_i + 4t + 2t^{\nu 5} f^i\gamma_i\gamma_\nu + t^5 f^i\gamma_i = 0 ,$$

where

$$f^i = g_{5j} g^{ji} .$$

Comparing the coefficients of each base of sedenion in the above, we have

$$t^{\lambda\mu} = t = 0 ;$$

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(1) From (2.4) we have

$$\gamma_{(i} W \gamma_{i)} = W^{-1} g_{ii} ,$$

$$\therefore \qquad \gamma_i(\gamma_i W \gamma_i) = \gamma_i W^{-1} g_{ii} ,$$

$$\therefore \qquad W\gamma_i = \gamma_i W^{-1} \qquad (\text{if } g_{ii} \neq 0)$$

and conversely, if this condition is satisfied, (2.4) holds, i.e.

$$\gamma_{(i} W \gamma_{j)} = W^{-1} g_{ij} .$$

substituting this into (2.8) we have

$$t^\nu g_{\nu i} = 0\,.$$

Therefore the infinitesimal transformation $(I+\omega)$ which satisfies (2.7) is determined uniquely, except for the multiplier, as follows:

$$\omega = t^\nu \gamma_\nu\,,$$

where

$$t^\nu g_{\nu i} = 0\,.$$

So that the finite form for W is obtained as

$$W = e^{t^\nu \gamma_\nu}$$

and

$$\Omega = S e^{t^\nu \gamma_\nu}\,. \tag{2.9}$$

So we have the following result: *The most general transformation which makes* $ds\Psi = 0$ *invariant is*

$$\left.\begin{aligned} {}'x^i &= f_i(x^1, \ldots, x^4)\,, \\ {}'\Psi &= S\Psi \\ {}'\gamma_i &= \frac{\partial x^j}{\partial' x^i} S e^{t^\nu \gamma_\nu} \gamma_j S^{-1} \end{aligned}\right\} \tag{2.10}$$

where S *is any* 4–4 *matrix and* t^ν *is given by* $t^\nu g_{\nu i} = 0$.

Specially, if we consider the transformation

$$\left.\begin{aligned} {}'x^i &= x^i \\ {}'\Psi &= \Psi \end{aligned}\right\} \tag{2.11}$$

we have from (2.9)

$${}'\gamma_i = e^{t^\nu \gamma_\nu} \gamma_i\,.$$

We call this transformation the *gauge transformation.* We can prove that this gauge transformation is equivalent to the following transformation which also makes $ds\Psi = 0$ invariant

$$\left.\begin{aligned} {}'x^i &= x^i \\ {}'\Psi &= R\Psi \\ {}'\gamma_i &= \gamma_i \end{aligned}\right\} \tag{2.12}$$

where

$$R = e^{-t^\nu \gamma_\nu}.$$

Proof.

Since (2.12) makes $ds\psi = 0$ invariant, we have

$$\gamma_i = Re^{v^\nu \gamma_\nu}\gamma_i R^{-1},$$

where

$$v^\nu g_{\nu i} = 0,$$

and since

$$e^{v^\nu \gamma_\nu}\gamma_i = \gamma_i = \gamma_i(e^{v^\nu \gamma_\nu})^{-1}$$

it must be that

$$\gamma_i = Re^{\frac{1}{2}v^\nu \gamma_\nu}\gamma_i e^{-\frac{1}{2}v^\nu \gamma_\nu}R^{-1};$$

so we have

$$R = e^{-\frac{1}{2}v^\nu \gamma_\nu}.$$

If we put $\frac{1}{2}v^\nu = t^\nu$ we have

$$'ds'\psi = '\gamma_i' dx^{i'}\psi = \gamma_i e^{-t^\nu \gamma_\nu} dx^i \psi = e^{t^\nu \gamma_\nu}\gamma_i dx^i \psi,$$

hence

$$'\gamma_i' dx^{i'}\psi = e^{t^\nu \gamma_\nu}\gamma_i dx^i \psi.$$

Conversely from (2.12) we can introduce (2.11), therefore (2.11) and (2.12) are equivalent.

Transformation (2.10) can be analysed into the following four sets:

$$\left.\begin{aligned} 'x^i &= f_i(x^1, \ldots, x^4) \\ '\psi &= \psi \\ '\gamma_i &= \frac{\partial x^j}{\partial' x^i}\gamma_j \end{aligned}\right\}, \qquad \left.\begin{aligned} 'x^i &= x^i \\ '\psi &= S\psi \\ '\gamma_i &= S\gamma_i S^{-1} \end{aligned}\right\},$$

$$\left.\begin{aligned} 'x^i &= x^i \\ '\psi &= \psi \\ \gamma_i &= e^{t^\nu \gamma_\nu}\gamma_i \end{aligned}\right\}, \qquad \left.\begin{aligned} 'x^i &= x^i \\ '\psi &= e^{-t^\nu \gamma_\nu}\psi \\ '\gamma_i &= \gamma_i \end{aligned}\right\}.$$

But as the third and fourth transformations are equivalent, we have the following result.

The transformation which makes $ds\psi = 0$ *invariant is composed of the following three transformations:*

Coordinate transformation

$$\left.\begin{aligned} 'x^i &= f_i(x^1, \ldots\ldots, x^4) \\ '\Psi &= \Psi \\ '\gamma_i &= \frac{\partial x^j}{\partial' x^i}\gamma_j \\ 'g_{ij} &= \frac{\partial x^k}{\partial' x^i}\frac{\partial x^l}{\partial' x^j}g_{kl} \end{aligned}\right\} \ldots\ldots \textit{C-transformation.}$$

Spin transformation

$$\left.\begin{aligned} 'x^i &= x^i \\ '\Psi &= S\Psi \\ '\gamma_i &= S\gamma_i S^{-1} \\ 'g_{ij} &= g_{ij} \end{aligned}\right\} \ldots\ldots \textit{S-transformation.}$$

Gauge transformation

$$\left.\begin{aligned} 'x^i &= x^i \\ '\Psi &= \Psi \\ '\gamma_i &= e^{t^\nu\gamma_\nu}\gamma_i \\ 'g_{ij} &= g_{ij} \end{aligned}\right\} \ldots\ldots \textit{G-transformation.}$$

The expression for the G-transformation can be written in a more convenient form, namely,

$$(t^\nu\gamma_\nu)^2 = t^\nu t^\mu\gamma_\nu\gamma_\mu = t^\nu t^\mu g_{\nu\mu} = (t^5)^2 g_{55}\,,$$

$$(t^\nu\gamma_\nu)^3 = (t^5)^2 g_{55} t^\nu\gamma_\nu\,,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

hence

$$e^{t^\nu\gamma_\nu} = I\cosh w + A^\nu\gamma_\nu\sinh w\,, \tag{2.13}$$

where

$$A^\nu = \frac{t^\nu}{t^5\sqrt{g_{55}}}\,.$$

Therefore G-transformation becomes

$$\left.\begin{aligned} {}'x^i &= x^i \\ {}'\psi &= \psi \\ {}'\gamma_i &= (I\cosh w + A^\nu\gamma_\nu \sinh w) \\ {}'g_{ij} &= g_{ij} \end{aligned}\right\},$$

where

$$A^\nu A^\mu g_{\nu\mu} = 1\,.$$

Putting the G and S-transformations together we call them the ψ-*transformation.*

Special case I. When the transformation makes $ds\psi$ invariant it becomes

$$\left.\begin{aligned} {}'x^i &= f_i(x^1, \ldots\ldots, x^4) \\ {}'\psi &= e^{-t^\nu\gamma_\nu}\psi \\ {}'\gamma_i &= \frac{\partial x^j}{\partial' x^i}\gamma_j e^{t^\nu\gamma_\nu} \\ {}'g_{ij} &= \frac{\partial x^k}{\partial' x^i}\,\frac{\partial x^l}{\partial' x^j}\, g_{kl} \end{aligned}\right\}.$$

Special case II. When we choose γ_5 such that $g_{5\lambda} = \delta_{5\lambda}$, we have

$$t^i = 0 \qquad \text{and} \qquad t^5 = \text{any}\,;$$

therefore the gauge transformation becomes

$$\left.\begin{aligned} {}'x^i &= x^i \\ {}'\psi &= \psi \\ {}'\gamma_i &= e^{t^5\gamma_5}\gamma_i \end{aligned}\right\}.$$

In this section we have based our calculation on the invariancy of $ds\psi = 0$, *though without logical signification. However the transformations obtained under this assumption are equivalent to those in the ordinary spinor calculus except for the gauge transformation. But in the spinor calculus the gauge transformation would also have been* *treated since the transformation must have contained all those which make g_{ij} invariant. Thus the assumption of the invariancy of $ds\psi = 0$ has an important significance.*

§ 3. Lorentz transformation.

When $\gamma_i = \mathring{\gamma}_i$ we consider the case where $'\gamma_i = \mathring{\gamma}_i$ in the transformation which makes $ds\psi = 0$ invariant.

Since the transformation (2.10) is independent of the choice of $g_{5\lambda}$ while γ_i or g_{ij} is fixed, $e^{t^\nu \gamma_\nu}$ is unaltered whatever may be the value of γ_5. Therefore we can assume, without losing generality, that $g_{5i} = 0$ and $g_{55} =$ any. So we have from (2.10)

$$\frac{\partial' x^i}{\partial x^j} \mathring{\gamma}_i = S e^{t^5 \gamma_5} \mathring{\gamma}_j S^{-1} .$$

Specially, if we consider all the transformations (C, G, S-transformations) to be infinitesimal, the above equation becomes

$$(\delta^i_j + p^i_j)\mathring{\gamma}_i = (I+\sigma)(I+t^5\mathring{\gamma}_5)(I-\sigma) .$$

Expanding σ in sedenion

$$\sigma = S^{\lambda\mu}\mathring{\gamma}_\lambda \mathring{\gamma}_\mu + S^\lambda \mathring{\gamma}_\lambda + SI \qquad (S^{\lambda\mu} = -S^{\mu\lambda}) ,$$

and substituting this into the above, we have, comparing the coefficients of the bases of sedenion,

$$S^{5i} = 0 , \quad S^j = 0 , \quad S^5 = \frac{-t^5}{2} \qquad \text{and} \qquad p^i_j = 4S^{ij} ,$$

so

$$\sigma = \frac{1}{4} p^i_j \mathring{\gamma}_i \mathring{\gamma}_j - \frac{t^5}{2} \mathring{\gamma}_5 + SI \qquad \text{and} \qquad p^i_j = -p^j_i .$$

Therefore the S transformation is divided into two linear transformations

$$(\psi_1, \psi_2) \longrightarrow ('\psi_1, '\psi_2) ,$$

$$(\psi_3, \psi_4) \longrightarrow ('\psi_3, '\psi_4) ,$$

which may both be arbitrary and from the fact $p^i_j = -p^j_i$ we know that the C-transformation is a rotation; and conversely.

Specially, when S is a constant matrix the coordinate transformation becomes Lorentzian and this relation between the Lorentz transformation and the ψ-transformation is a well known fact.[1]

§4. The displacements which make $ds\psi = 0$ invariant.

As stated in §2, we have considered the transformations which make $ds\psi = 0$ invariant. So *it is most natural to consider the parallel displacements which make* $ds\psi = 0$ *invariant;* this corresponds to Weyl's parallel displacement in ordinary space.

Now we will investigate such parallel displacement.

γ_i is a function of x^k defined at each point in x space, and ψ is also a function given as a solution of a certain partial differential equation — "*the unknown Dirac equation.*"[2]

Let two vectors $\overline{dx^i}$ and dx^i at any two consecutive points x^i and $x^i+\delta x^i$ be parallel to each other; then, for the function sets $(\gamma_i, \psi, \overline{dx^i})_x$ and $(\gamma_i, \psi, dx^j)_{x+\delta x}$ at the two points, $ds\psi = 0$ must be invariant. So we have the following relation

$$(\gamma_i\overline{dx^i}\psi)_x = (I+\Lambda_m\delta x^m)(\gamma_i dx^i\psi)_{x+dx}\,, \tag{4.1}$$

and

$$\overline{dx^i} = dx^i + \Gamma^i_{jk}dx^j\delta x^k\,, \tag{4.2}$$

where Λ_m is a certain 4–4 matrix and Γ^i_{jk} is a function of x^l which is a general coefficient of connection in x space.

Expanding (4.1), substituting (4.2) and neglecting the second order of δx^i, we have

$$\left\{\left(\gamma_i\Gamma^i_{lm} - \Lambda_m\gamma_l - \frac{\partial\gamma_l}{\partial x^m}\right)\psi - \gamma_l\frac{\partial\psi}{\partial x^m}\right\}dx^l\delta x^m = 0\,.$$

This must hold for all dx^i, δx^i; so we have

$$\gamma_l\frac{\partial\psi}{\partial x^m} = \left(\gamma_i\Gamma^i_{lm} - \Lambda_m\gamma_l - \frac{\partial\gamma_l}{\partial x^m}\right)\psi\,. \tag{4.3}$$

Multiplying the above by γ_l

$$\frac{\partial\psi}{\partial x^m} = \frac{1}{g_{ll}}\left(\gamma_l\gamma_i\Gamma^i_{lm} - \gamma_l\Lambda_m\gamma_l - \gamma_l\frac{\partial\gamma_l}{\partial x^m}\right)\psi\,, \tag{4.4}$$

where we do not sum by l.

(1) H. Weyl, *Gruppen Theorie und Quanten Mechanik,* (1928), 110.

(2) Prof. Y. Mimura suggested this idea in his paper; Y. Mimura: ibid. 104–105.

Next we shall obtain the condition that the right hand side of (4.4) must be independent of l. For this purpose we shall assume that the coefficient of ψ in the right hand side of (4.4) is independent of l for all values of ψ. — But in fact, since ψ may be a solution of a certain partial differential equation dependent of γ_l, the coefficient of ψ might not be considered as independent of ψ, and as this case presents considerable difficulty we will leave it for future research;[(1)] and in the last section of this paper we shall discuss only some possible cases.[(2)]

Expand Λ_m in sedenion

$$\Lambda_m = L_m^{\lambda\mu}\gamma_\lambda\gamma_\mu + L_m^\lambda\gamma_\lambda + L_m I, \qquad L_m^{\lambda\mu} = -L_m^{\mu\lambda}, \tag{4.5}$$

and consider following identity[(3)]

$$\frac{\partial\gamma_l}{\partial x^m} = \{{}_{lm}^{\sigma}\}\gamma_\sigma + \Gamma_m\gamma_l - \gamma_l\Gamma_m, \tag{4.6}$$

where Γ_m is a 4–4 matrix and may therefore have the following expansion

$$\Gamma_m = C_m^{\lambda\mu}\gamma_\lambda\gamma_\mu + C_m^\lambda\gamma_\lambda \qquad C_m^{\lambda\mu} = -C_m^{\mu\lambda}. \tag{4.7}$$

If we substitute (4.5), (4.6) into (4.4) and calculate the condition that the coefficient of the ψ in the right hand side of the equation so obtained must be independent of suffix l, we have the following relations after some calculation[(4)]

$$\begin{cases} (L_m^\lambda + C_m^\lambda)g_{\lambda l} = 0\,, & (4.8)\\ \Gamma_{lm}^j = \{{}_{lm}^{\,j}\} + 4(L_m^{j\mu} + C_m^{j\mu})g_{\mu l} + \delta_l^j R_m\,, & (4.9)\\ 0 = \{{}_{lm}^{\,5}\} + 4(L_m^{5\mu} + C_m^{5\mu})g_{\mu l}\,, & (4.10)\end{cases}$$

where R_m = any covariant vector.

Substituting the above relations into (4.4) we immediately have the following equation

$$\frac{\partial\psi}{\partial x^m} = \{(-L_m^{\lambda\mu}\gamma_\lambda\gamma_\mu + L_m^\lambda\gamma_\lambda - L_m I) + R_m I - 2C_m^\lambda\gamma_\lambda\}\psi\,; \tag{4.11}$$

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(1) I succeeded the treatment of this case after having presented this paper. I will publish it in the next volume of this Journal.

(2) See § 9, p. 174.

(3) Pauli, *loc. cit.* 356.

(4) See **Note 1** at the end of this paper, p. 177.

here if we put

$$L^{\lambda}_{m}+C^{\lambda}_{m} = T^{\lambda}_{m},$$

(4.11) becomes as follows

$$\frac{\partial\psi}{\partial x^m} = (-\Lambda_m+2T^{\lambda}_{m}\gamma_\lambda+R_m I)\psi. \tag{4.12}$$

Next we define the operator

$$\nabla_m = \frac{\partial}{\partial x^m}+\Lambda_m-2T^{\lambda}_{m}\gamma_\lambda.$$

From the fact that the parallelism between vectors $\overline{dx^i}$ and dx^i in x space must be invariant for all the fundamental transformations we can prove that this operator has invariant property for all the transformations which make $ds\psi = 0$ invariant. Namely,
By the C-transformation,(1)

$$'\nabla_m = \frac{\partial x^i}{\partial' x_m}\nabla_i,$$

By the S-transformation $'\psi = S\psi$, we have after calculation(2)

$$\left.\begin{aligned} &'\nabla_m = \frac{\partial}{\partial x^m}+'\Lambda_m-2'T^{\lambda}_{m}{}'\gamma_\lambda, \quad \text{i. e.} \quad '\nabla_m = S\nabla_m S^{-1}, \\ &\text{where} \\ &'\Lambda_m = S\Lambda_m S^{-1}+S\frac{\partial S^{-1}}{\partial x^m} \quad \text{and} \quad 'T^{\lambda}_{m} = T^{\lambda}_{m}. \end{aligned}\right\} \tag{4.13}$$

And by the G-transformation $'\gamma_i = W\gamma_i = I\cosh w+A^\nu\gamma_\nu \sinh w$, (3) we have

$$\left.\begin{aligned} &'\nabla_m = \frac{\partial}{\partial x^m}+'\Lambda_m-2'T^{\lambda}_{m}{}'\gamma_\lambda, \\ &\text{where} \\ &'\Lambda_m = W\Lambda_m W^{-1}+W\frac{\partial W^{-1}}{\partial x^m} = \Lambda_m-\frac{\partial w}{\partial x^m}A^\lambda\gamma_\lambda \\ &\text{and} \\ &'T^{\lambda}_{m} = T^{\lambda}_{m}-\frac{1}{2}\frac{\partial w}{\partial x^m}A^\lambda \end{aligned}\right\} \tag{4.14}$$

(1) See **Note 2,** p. 179.
(2) See **Note 3,** p. 179.
(3) See **Note 4,** p. 181.

consequently we have $'\nabla_m = \nabla_m$.

R_m is a convariant for the C-transformation and is invariant for ψ-transformation.[1]

Therefore, *we see that* (4.12) *or*

$$\nabla_m \psi = R_m I \psi \tag{4.15}$$

is an invariant equation for all C, G, S-transformation.

Consequently, *when we give the microscopic metric* γ_i, *the affine connection* Γ^i_{jk} *in the x space and* Λ_m, *the equation (4.15) may be considered as the partial differential equation satisfying by* ψ *and invariant for all C, G, S-transformations; therefore this is nothing but the required " unknown Dirac equation."*

§ 5. Some identities.

In this section for simplicity's sake we calculate formulas so that $g_{5\lambda} = \delta_{5\lambda}$.

Then (4.8), (4.9) becomes

$$L^i_m + C^i_m = 0\,, \tag{5.1}$$

$$\Gamma^i_{jk} = \{^{\,i}_{jk}\} + 4(L^{il}_k + C^{il}_k) g_{li} + R_k \delta^i_j\,. \tag{5.2}$$

1). After Schlödinger- Pauli's definition of covariant derivative of γ_i

$$\mathring{\nabla}_m \gamma_i = \frac{\partial \gamma_i}{\partial x^m} - \{^{\,j}_{im}\}\gamma_j - \Gamma_m \gamma_i + \gamma_i \Gamma_m\,,$$

we define the covariant derivative of γ_i by the operator ∇_m as follows:

$$\nabla_m \gamma_i = \frac{\partial \gamma_i}{\partial x^m} - \Gamma^j_{im}\gamma_j + (\Lambda_m - 2T^\lambda_m \gamma_\lambda)\gamma_i - \gamma_i(\Lambda_m - 2T^\lambda_m \gamma_\lambda)\,;$$

if we calculate the above by using the identity $\mathring{\nabla}_m \gamma_i \equiv 0$, and (5.1) and (5.2), we have

$$\nabla_m \gamma_i = -R_m \gamma_i - 2T^\lambda_m \gamma_{[\lambda}\gamma_{i]}\,, \tag{5.3}$$

which shows that this ∇_m is a invariant operator for all the C, G, S-transformation.

2). From (5.2) we have

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(1) See **Note 5, p. 183.**

and

$$\left.\begin{aligned} g^{ik}R_m &= M^{(k\cdot i)}_{\ \ m}\,, \\ 4(L^{ik}_m + C^{ik}_{m)}) &= M^{[k\cdot i]}_{\ \ m} \end{aligned}\right\}, \tag{5.4}$$

where

$$M_{km}{}^{i}{}_{\cdot}{}^{i} = \Gamma^i_{km} - \left\{{i \atop km}\right\}. \tag{5.5}$$

Specially, when the affine connection of x space is Weyl's case, we have from (5.5)

$$M^{i\cdot j}_m = \frac{1}{2}(Q^i\delta^j_m + Q_m g^{ij} - Q^j\delta^i_m)\,,^{(1)}$$

therefore

$$\left.\begin{aligned} Q_m &= 2R_m\,, \\ 8(L^{jl}_m + C^{jl}_{m)}) &= Q^l\delta^j_m - Q^j\delta^l_m \end{aligned}\right\}. \tag{5.6}$$

The fundamental equation for ψ then becomes

$$\frac{\partial\psi}{\partial x^m} = \left\{\Gamma_m + T^5_m\gamma_5 + \left(\frac{1}{2}Q_m - L_m\right)I\right\}\psi\,. \tag{5.7}$$

When γ_i is given Γ_m is determined, in addition to this, if we give Λ_m and R_m so as to satisfy (5.1), then the coefficient of affine connection of x space Γ^i_{jk} is determined. In Weyl's case it is possible to give R_m and Λ_m so as to satisfy (5.6); therefore the Weyl's connection Γ^i_{jk} is obtained from (5.2). So we have the following result.

The parallelism, of which $ds\,\psi = 0$ admits, contains the Weyl's parallelism.

3). In the case in which the x space has Riemannian connection[2] we have from (5.6)

$$R_m = 0\,,$$

and from (5.2)

$$L^{ij}_m + C^{ij}_m = 0\,,$$

$$L^i_m + C^i_m = 0\,,$$

therefore

$$\Lambda_m + \Gamma_m = T^5_m\gamma_5 + L_m I\,. \tag{5.8}$$

So the fundamental equation for ψ becomes

(1) J. A. Schouten, *Der Ricci Kalkül*, (1924), 75.
(2) In such case we say that the space is Riemannian.

$$\frac{\partial \psi}{\partial x^m} = (\Gamma_m + T^5_m \gamma_5 - L_m I)\psi . \tag{5.9}$$

Hence we have the following result. *In Riemannian space, for given γ_i, g_{ij} is determined by $g_{ij} \doteq \gamma_{(i}\gamma_{j)}$, and from the equation*

$$\frac{\partial \gamma_i}{\partial x^m} = \left\{{k \atop im}\right\}\gamma_k + \Gamma_m \gamma_i - \gamma_i \Gamma_m$$

Γ_m is determined uniquely except for the unit term; moreover γ_5 is obtained from

$$\gamma_{(5}\gamma_{\lambda)} = \delta_{5\lambda} ,$$

and then by giving T^5_m and L_m as any covariant vectors ψ can be obtained from (5.9).

4). When T^5_m in (5.9) is a gradient vector i. e.

$$T^5_m = \frac{1}{2}\frac{\partial w}{\partial x^m} ,$$

by the following G-transformation

$$W = I \cosh w + \gamma_5 \sinh w ,$$

$$'\gamma_i = W_i ,$$

we have

$$'T^5_m = 0 \qquad \text{(from (4.14))}.$$

When L_m is a gradient i. e.

$$L_m = -\frac{\partial \log \rho}{\partial x^m} ,$$

by the following S-transformation

$$\left.\begin{aligned} '\psi &= S\psi \\ '\gamma_i &= S\gamma_i S^{-1} \\ S &= \rho I \end{aligned}\right\} ,$$

we have

$$'L_m = 0 \qquad \text{(from (4.13))}.$$

Therefore *when T^5_m or L_m is a gradient we can after suitable G and S-transformations set off the term from the fundamental equation for ψ.*

5). When the parallel displacement makes $ds\psi$ invariant, we have $\Lambda_m = 0$; so (4.12) becomes

$$\frac{\partial\psi}{\partial x^m} = (2T^{\lambda}_{m}\gamma_{\lambda} + R_m I)\psi .$$

Adding to this, if the space is Riemannian, we have $R_m = 0$ from (5.6); so the above equation becomes

$$\frac{\partial\psi}{\partial x^m} = 2T^{\lambda}_{m}\gamma_{\lambda}\psi . \tag{5.10}$$

The condition of integrability is that

$$T^{\lambda}_{m} = \text{gradient vector in suffix } m,$$

therefore, after a suitable G-transformation we can get rid of the term from the equation (5.10), and we have

$$\frac{\partial\psi}{\partial x^m} = 0$$

which supplies us only a trivial solution $\psi = \text{const.}$

§ 6. The condition for integrability of the fundamental equation for ψ.

We shall now obtain the condition for integrability of the fundamental equation

$$\frac{\partial\psi}{\partial x^m} = (-\Lambda_m + 2T^{5}_{m}\gamma_5 + R_m I)\psi , \tag{6.1}$$

when $\gamma_i, \Gamma^i_{jk}, \Lambda_m$ are given.

From (4.5), (4.7), (4.8), and (4.9), we have

$$\Lambda_m = -\Gamma_m + \frac{1}{4}(\Gamma^i_{lm} - \{^{\,i}_{lm}\})g^{kl}\gamma_{[i}\gamma_{k]} + L_m I + T^5_{m}\gamma_5 ,$$

and substituting this into (6.1) we have

$$\frac{\partial\psi}{\partial x^m} = \left[\Gamma_m - \frac{1}{4}(\Gamma^i_{lm} - \{^{\,i}_{lm}\})g^{kl}\gamma_{[i}\gamma_{k]} + (R_m - L_m)I + T^5_{m}\gamma_5\right]. \tag{6.2}$$

If we write down the condition for integrability of the above

$$\frac{\partial^2 \psi}{\partial x^{[m} \partial x^{l]}} = 0,$$

and calculate[1] this by substituting (6.2) and (5.3), we have

$$\left[\frac{1}{4} R_{lmij} \gamma^{[i} \gamma^{j]} + t_{lm} \gamma_5 + f_{lm} I\right] \psi = 0, \tag{6.3}$$

where R_{lmij} is the curvature tensor derived from Γ^i_{jk} such that

$$R_{lmi}^{\cdot\cdot\cdot j} = \frac{\partial \Gamma^j_{il}}{\partial x^m} - \frac{\partial \Gamma^j_{im}}{\partial x^l} - \Gamma^j_{kl} \Gamma^k_{im} + \Gamma^j_{km} \Gamma^k_{il},$$

$$t_{lm} = \frac{\partial T^5_l}{\partial x_m} - \frac{\partial T^5_m}{\partial x^l},$$

and

$$f_{lm} = \frac{\partial}{\partial x^m}(R_l - L_l) - \frac{\partial}{\partial x^l}(R_m - L_m).$$

Again if we differentiate (6.3) by x^r and calculate[2] this by substituting (5.3), (6.2), (6.3) etc. we have

$$\left[\frac{1}{4} R_{lmij,r} \gamma^{[i} \gamma^{j]} + t_{lm,r} \gamma_5 + f_{lm,r} I\right] \psi = 0, \tag{6.4}$$

where $T^{\cdots}_{\cdots,r}$ expresses the covariant derivative of $T^{\cdots}_{\cdots}$ with respect to the coefficient of connection $\Gamma^i_{jk} - \delta^i_j R_k$. Similarly

$$\left[\frac{1}{4} R_{lmij,rs\ldots} \gamma^{[i} \gamma^{j]} + t_{lm,rs\ldots} \gamma_5 + f_{lm,rs\ldots} I\right] \psi = 0. \tag{6.5}$$

Therefore we have the following result. *The condition for integrability of (6.1) is that (6.3), (6.4) and (6.5) are compatible.*

We will now consider this condition for integrability precisely when the macroscopic metric g_{ij} is given originally.

In this case, from the relation

$$\gamma_{(i} \gamma_{j)} = g_{ij}, \tag{6.6}$$

we can find γ_i in the following form

(1) See **Note 6**, p. 184.
(2) See **Note 7**, p. 185.

$$\gamma_i = U h^{j}_{i} \overset{\circ}{\gamma}_j U^{-1} , \tag{6.7}$$

and the

$$g_{ij} = \sum_{a} h^{a}_{i} h^{a}_{j} .$$

Further if we put

$$g_{5\lambda} = \delta_{5\lambda} ,$$

then

$$\gamma_5 = U \overset{\circ}{\gamma}_5 U^{-1} .$$

Substituting (6.7) into (6.3), (6.4), etc., we have

$$\left.\begin{aligned}
&\Big(\frac{1}{4} R_{ilm}{}^{ij} h^{p}_{[i} h^{q}_{j]} \overset{\circ}{\gamma}_p \overset{\circ}{\gamma}_q + f_{lm} I + t_{lm} \overset{\circ}{\gamma}_5\Big)\overline{\Psi} = 0 ,\\
&\Big(\frac{1}{4} R_{ilm}{}^{ij}{}_{,r} h^{p}_{[i} h^{q}_{j]} \overset{\circ}{\gamma}_p \overset{\circ}{\gamma}_q + f_{lm,r} I + t_{lm,r} \overset{\circ}{\gamma}_5\Big)\overline{\Psi} = 0 ,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ,
\end{aligned}\right\} \tag{6.8}$$

where

$$\overline{\Psi} = U^{-1}\Psi .$$

If we rewrite (6.8) in actual form from the matrix form, we have

$$\left.\begin{aligned}
&(i\overset{3}{k}_{lm} - i\overset{4}{k}_{lm} + f_{lm} + t_{lm})\overline{\Psi}_1 + (i\overset{1}{k}_{lm} - \overset{2}{k}_{lm} - \overset{5}{k}_{lm} - i\overset{6}{k}_{lm})\overline{\Psi}_2 = 0 ,\\
&(i\overset{1}{k}_{lm} + \overset{2}{k}_{lm} + \overset{5}{k}_{lm} - i\overset{6}{k}_{lm})\overline{\Psi}_1 + (-i\overset{3}{k}_{lm} + i\overset{4}{k}_{lm} + f_{lm} + t_{lm})\overline{\Psi}_2 = 0 ,\\
&(-i\overset{3}{k}_{lm} - i\overset{4}{k}_{lm} + f_{lm} - t_{lm})\overline{\Psi}_3 + (-i\overset{1}{k}_{lm} + \overset{2}{k}_{lm} - \overset{5}{k}_{lm} - i\overset{6}{k}_{lm})\overline{\Psi}_4 = 0 ,\\
&(-i\overset{1}{k}_{lm} - \overset{2}{k}_{lm} + \overset{5}{k}_{lm} - i\overset{6}{k}_{lm})\overline{\Psi}_3 + (i\overset{3}{k}_{lm} + i\overset{4}{k}_{lm} + f_{lm} - t_{lm})\overline{\Psi}_4 = 0 ,\\
&(i\overset{3}{k}_{lm,r} - i\overset{4}{k}_{lm,r} + f_{lm,r} + t_{lm,r})\overline{\Psi}_1 + (i\overset{1}{k}_{lm,r} - \overset{2}{k}_{lm,r} - \overset{5}{k}_{lm,r} - i\overset{6}{k}_{lm,r})\overline{\Psi}_2 = 0 ,\\
&(i\overset{1}{k}_{lm,r} + \overset{2}{k}_{lm,r} + \overset{5}{k}_{lm,r} - i\overset{6}{k}_{lm,r})\overline{\Psi}_1 + (-i\overset{3}{k}_{lm,r} + i\overset{4}{k}_{lm,r} + f_{lm,r}\\
&\qquad\qquad + t_{lm,r})\overline{\Psi}_2 = 0 ,\\
&(-i\overset{3}{k}_{lm,r} - i\overset{4}{k}_{lm,r} + f_{lm,r} - t_{lm,r})\overline{\Psi}_3 + (-i\overset{1}{k}_{lm,r} + \overset{2}{k}_{lm,r} - \overset{5}{k}_{lm,r}\\
&\qquad\qquad - i\overset{6}{k}_{lm,r})\overline{\Psi}_4 = 0 ,\\
&(-i\overset{1}{k}_{lm,r} - \overset{2}{k}_{lm,r} + \overset{5}{k}_{lm,r} - i\overset{6}{k}_{lm,r})\overline{\Psi}_3 + (i\overset{3}{k}_{lm,r} + i\overset{4}{k}_{lm,r} + f_{lm,r}\\
&\qquad\qquad - t_{lm,r})\overline{\Psi}_4 = 0 ,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ,
\end{aligned}\right\} \tag{6.9}$$

where

$$2\overset{1}{k}_{lm} = R_{lm}{}^{ij}h^1_{[i}h^2_{j]}\,,$$

$$2\overset{2}{k}_{lm} = R_{lm}{}^{ij}h^1_{[i}h^3_{j]}\,,$$

$$2\overset{3}{k}_{lm} = R_{lm}{}^{ij}h^1_{[i}h^4_{j]}\,,$$

$$2\overset{4}{k}_{lm} = R_{lm}{}^{ij}h^2_{[i}h^3_{j]}\,,$$

$$2\overset{5}{k}_{lm} = R_{lm}{}^{ij}k^2_{[i}h^4_{j]}\,,$$

$$2\overset{6}{k}_{lm} = R_{lm}{}^{ij}h^3_{[i}h^4_{j]}\,.$$

Therefore *as the condition for which* (6.9) *may have a non vanishing solution* $\overline{\Psi}$ *we have the following equations:*

either

$$\left.\begin{aligned}
&(\overset{1}{k}_{lm}-\overset{6}{k}_{lm})^2+(\overset{2}{k}_{lm}+\overset{5}{k}_{lm})^2+(\overset{3}{k}_{lm}-\overset{4}{k}_{lm})^2+(t_{lm}+f_{lm})^2=0\,,\\
&(\overset{1}{k}_{lm,\,rs\ldots}-\overset{6}{k}_{lm,\,rs\ldots})^2+(\overset{2}{k}_{lm,\,rs\ldots}+\overset{5}{k}_{lm,\,rs\ldots})^2+(\overset{3}{k}_{lm,\,rs\ldots}-\overset{4}{k}_{lm,\,rs\ldots})^2\\
&\qquad\qquad\qquad\qquad+(t_{lm,\,rs\ldots}+f_{lm,\,rs\ldots})^2=0\,,\\
&\frac{i\overset{3}{k}_{st}-i\overset{4}{k}_{st}+f_{st}+t_{st}}{i\overset{1}{k}_{st}-\overset{2}{k}_{st}-\overset{5}{k}_{st}-i\overset{6}{k}_{st}}=\frac{i\overset{3}{k}_{pq,\,r\ldots}-i\overset{4}{k}_{pq,\,r\ldots}+f_{pq,\,r\ldots}+t_{pq,\,r\ldots}}{i\overset{1}{k}_{pq,\,r\ldots}-\overset{2}{k}_{pq,\,r\ldots}-\overset{5}{k}_{pq,\,r\ldots}-i\overset{6}{k}_{pq,\,r\ldots}}
\end{aligned}\right\}\ (6.10)\,,$$

or

$$\left.\begin{aligned}
&(\overset{1}{k}_{lm}+\overset{6}{k}_{lm})^2+(\overset{2}{k}_{lm}-\overset{5}{k}_{lm})^2+(\overset{3}{k}_{lm}+\overset{4}{k}_{lm})^2+(t_{lm}-f_{lm})^2=0,\\
&(\overset{1}{k}_{lm,\,rs\ldots}+\overset{6}{k}_{lm,\,rs\ldots})^2+(\overset{2}{k}_{lm,\,rs\ldots}-\overset{5}{k}_{lm,\,rs\ldots})^2+(\overset{3}{k}_{lm,\,rs\ldots}+\overset{4}{k}_{lm,\,rs\ldots})^2\\
&\qquad\qquad\qquad\qquad+(t_{lm,\,rs\ldots}-f_{lm,\,rs\ldots})^2=0\,,\\
&\frac{i\overset{3}{k}_{st}+i\overset{4}{k}_{st}-f_{st}+t_{st}}{i\overset{1}{k}_{st}-\overset{2}{k}_{st}+\overset{5}{k}_{st}+i\overset{6}{k}_{st}}=\frac{i\overset{3}{k}_{pq,\,r\ldots}+i\overset{4}{k}_{pq,\,r\ldots}-f_{pq,\,r\ldots}+t_{pq,\,r\ldots}}{i\overset{1}{k}_{pq,\,r\ldots}-\overset{2}{k}_{pq,\,r\ldots}+\overset{5}{k}_{pq,\,r\ldots}+i\overset{6}{k}_{pq,\,r\ldots}}
\end{aligned}\right\}\ (6.11)\,.$$

§ 7. The condition of integrability in the simplest and important cases.

Before treating the general case,[1] we shall in this section consider the following simplest case which is significant in physical application. Namely, we assume, as in ordinary Riemannian geometry, that the coordinates x are all real, that the expression for the metric of the space $ds^2 = g_{ij}dx^i dx^j$ is a positive definite form and that $(f_{lm}+t_{lm})$ in (6.9) is real, but Γ^i_{jk} is most general.

Under this assumption we will express the condition (6.10) and (6.11) in terms of g_{ij} and its derivatives only.

From the assumption that $g_{ij}dx^i dx^j$ is positive definite h^i_j in (6.7) can be taken all real from the equation

$$g_{ij} = \sum_a h^a_i h^a_j$$

and $R_{lm}{}^{ij}$ therefore all become real. Thus $\overset{1}{k}_{lm}, \ldots, \overset{6}{k}_{lm}$ and $(t_{lm}+f_{lm})$ are real, so we have, from (6.10),

$$\overset{1}{k}_{lm} = \overset{6}{k}_{lm}, \quad \overset{2}{k}_{lm} = -\overset{5}{k}_{lm}, \quad \overset{3}{k}_{lm} = \overset{4}{k}_{lm} \quad \text{and} \quad t_{lm}+f_{lm} = 0. \tag{7.1}$$

When (7.1) holds, the first and second equations of (6.9) are satisfied for all $\overline{\psi}_1$, $\overline{\psi}_2$ identically, and from the third and fourth equation of (6.9) it must be that $\overline{\psi}_3 = \overline{\psi}_4 = 0$. In this case the remaining equations of (6.9) are all satisfied by such $\overline{\psi}$.

Therefore the equation for $\overline{\psi}$

$$\frac{\partial \overline{\psi}}{\partial x^m} = (-\overline{\Lambda}_m + 2\overline{T}^5_m \gamma_5 + \overline{R}_m I)\overline{\psi}\,,^{(2)} \tag{7.2}$$

which is equivalent to the fundamental equation for ψ

$$\frac{\partial \psi}{\partial x^m} = (-\Lambda_m + 2T^5_m \gamma_5 + R_m I)\psi \tag{7.3}$$

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(1) See § 8, p. 172.

(2) $\overline{\Lambda}_m = U^{-1}\Lambda_m U + \dfrac{\partial U^{-1}}{\partial x^m} U,\ \overline{T}^5_m = T^5_m,\ \overline{R}_m = R_m.$

is completely integrable for $\overline{\Psi}_1$, $\overline{\Psi}_2$ if we put $\overline{\Psi}_3 = \overline{\Psi}_4 = 0$, that is, (7.2) has a solution of the form

$$\overline{\Psi} = \begin{pmatrix} \overline{\Psi}_1 \\ \overline{\Psi}_2 \\ 0 \\ 0 \end{pmatrix},$$

therefore (7.3) is integrable.

Similarly from (6.11) we have another condition for integrability

$$\overset{1}{k}_{lm} = -\overset{6}{k}_{lm}, \ \overset{2}{k}_{lm} = \overset{5}{k}_{lm}, \ \overset{3}{k}_{lm} = -\overset{4}{k}_{lm} \quad \text{and} \quad f_{lm} = t_{lm}, \tag{7.4}$$

under which (7.2) is integrable, the solution being of the form

$$\overline{\Psi} = \begin{pmatrix} 0 \\ 0 \\ \overline{\Psi}_3 \\ \overline{\Psi}_4 \end{pmatrix}.$$

(The conditions (7.1) and (7.4) are not compatible unless $k_{lm} = 0$ —the space is euclidean.)

We shall now write the first three equations of (7.1) in more concrete forms.

Rewriting (7.1) we have

$$R_{lm}{}^{\cdot\cdot ij} h^1_{[i} h^2_{j]} = R_{lm}{}^{\cdot\cdot ij} h^3_{[i} h^4_{j]},$$

$$R_{lm}{}^{\cdot\cdot ij} h^1_{[i} h^3_{j]} = -R_{lm}{}^{\cdot\cdot ij} h^2_{[i} h^4_{j]},$$

$$R_{lm}{}^{\cdot\cdot ij} h^1_{[i} h^4_{j]} = R_{lm}{}^{\cdot\cdot ij} h^2_{[i} h^3_{j]},$$

and putting the above together in one equation,

$$R_{lm}{}^{\cdot\cdot ij} h^a_{[i} h^b_{j]} = ((a\,b\,c\,d))\, R_{lm}{}^{\cdot\cdot ij} h^c_{[i} h^d_{j]}, \tag{7.5}$$

(here we do not sum by c and d)

where, a, b, c, d is a permutation of 1, 2, 3, 4, and the expression $((a\,b\,c\,d))$ has the value 1 or -1 according as the number of the inversions is even or odd.

If we multiply (7.5) by $((a\,b\,c\,d))\, h^c_{[s} h^d_{t]}$, and take the summation by a, b, c, d and calculate[1] it, we have

$$\frac{\sqrt{\Delta}}{2}\epsilon_{stpq}R_{lm}^{\cdot\cdot\,pq} = R_{lm[st]}, \tag{7.6}$$

where

$$\Delta = \begin{vmatrix} g_{11} & \cdot & \cdot & g_{14} \\ \cdot & \cdot & \cdot & \cdot \\ g_{41} & \cdot & \cdot & g_{44} \end{vmatrix}.$$

Conversely, from (7.6) we can introduce the first three equations of (7.1) by reversing the procedure. Therefore (7.6) is one of the conditions of integrability of the fundamental equation (6.1). Similarly from (7.4) we have another condition of integrability

$$\frac{\sqrt{\Delta}}{2}\epsilon_{stpq}R_{lm}^{\cdot\cdot\,pq} = -R_{lm[st]}.$$

So we have the following result. *When $ds^2 = g_{ij}dx^i dx^j$ is positive definite and $f_{lm} \pm t_{lm}$ is real, the condition of integrability of the fundamental equation for ψ:*

$$\frac{\partial \psi}{\partial x^m} = (-\Lambda_m + 2T^5_m \gamma_5 + R_m I)\psi$$

is that

$$\left.\begin{aligned} &\frac{\sqrt{\Delta}}{2}\epsilon_{stpq}R_{lm}^{\cdot\cdot\,pq} = \pm R_{lm[st]}, \\ \textit{and}\quad & f_{lm} \pm t_{lm} = 0 \end{aligned}\right\}. \tag{7.7}$$

Specially, *when the space is Riemannian, i.e.*

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\},$$

then the fundamental equation for ψ becomes

$$\frac{\partial \psi}{\partial x^m} = (\Gamma_m + T^5_m \gamma_5 - L_m I)\psi \;^{(2)}$$

(1) See **Note 8,** p. 186.
(2) See p. 164, (5.9).

and the condition of integrability (7.7) *becomes then*

$$\frac{\sqrt{\Delta}}{2}\epsilon_{stpq}K_{lm}^{\cdot\cdot pq} = \pm K_{lmst} \tag{7.8}$$

$$f_{lm} \pm t_{lm} = 0 ,$$

where K_{lmpq} *is the Riemannian curvature tensor with respect to* g_{ij}.

If we contract t and l in (7.8) we have

$$\frac{\sqrt{\Delta}}{2}\epsilon_{stpq}K^{t\cdot pq}_{\cdot m} = \pm K_{ms} ,$$

or

$$\frac{\sqrt{\Delta}}{2}\epsilon_{tpqs}K_{m}^{\cdot[tpq]} = \pm K_{ms} ,^{(1)}$$

or

$$0 = K_{ms} ,$$

therefore we have the Einstein's law:

$$G_{ij} = 0 .$$

So we have the result: *When the space is Riemannian, the fundamental equation for* ψ *is*

$$\frac{\partial\psi}{\partial x^m} = (\Gamma_m + T^5_m\gamma_5 - L_m I)\psi ,$$

and the condition of integrability is (7.8); *and the space is then necessarily an Einstein's gravitational space.*

§ 8. The condition of integrability in general case.

In the previous section we have obtained the condition of integrability of the fundamental equation in terms of g_{ij} under the special assumption that ds^2 is positive definite. In this section we will find the expression of the condition of integrability in terms of g_{ij} in general case.

Rewriting the first equation of (6.10), we have, after a little calculation,

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(1) J. A. Schouten. *Der Ricci Kalkül,* (1924), 75.

$$R_{pq}^{\cdot\cdot}{}^{[ij]}R_{pq}^{\cdot\cdot}{}^{[lm]}\left\{\frac{1}{2}\sum_b h_i^b h_l^b \sum_a h_j^a h_m^a - 2(h_i^1h_j^2h_l^3h_m^4 + h_i^3h_j^1h_l^2h_m^4 + h_i^1h_j^4h_l^2h_m^3)\right\} + (f_{pq}+t_{pq})^2 = 0\,.$$

By a process similar to that used in obtaining (7.6) from (7.5) we have the following equation from the above

$$R_{pq}^{\cdot\cdot}{}^{ij}R_{pq}^{\cdot\cdot}{}^{lm}\epsilon_{ijlm}\sqrt{\Delta} = 2R_{pqij}R_{pq}^{\cdot\cdot}{}^{ij} + 4(f_{pq}+t_{pq})^2\,. \qquad (8.1)$$

Similarly from the second equation of (6.10), we have

$$R_{pq}^{\cdot\cdot}{}^{ij}{}_{,r..}R_{pq}^{\cdot\cdot}{}^{lm}{}_{,r..}\epsilon_{ijlm}\sqrt{\Delta} = 2R_{pqij,r...}R_{pq}^{\cdot\cdot}{}^{ij}{}_{,r..} + 4(f_{pq,r...}+t_{pq,r...})^2\,. \qquad (8.2)$$

The third equation of (6.10) can be written in the following form after some calculation[1]

$$\sqrt{\Delta}\,\epsilon_{abcd}\{2R_{st}^{\cdot\cdot}{}^{l[a}R_{pql}^{\cdot\cdot\cdot}{}^{b]}{}_{,r..} - i(f_{st}+t_{st})R_{pq}^{\cdot\cdot}{}^{[ab]}{}_{,r..} + (f_{pq,r...}+t_{pq,r...})R_{st}^{\cdot\cdot}{}^{[ba]})\}$$
$$= 2\left\{2R_{st\cdot[c}^{\cdot\cdot l}R_{|pql|p],r...} - i\big((f_{st}+t_{st})R_{pq[cd],r..} + (f_{pq,r...}+{}_{pq,r...})R_{st[cd]}\big)\right\}. \qquad (8.3)$$

Corresponding to the above, the condition (6.11) is written as

$$-R_{pq}^{\cdot\cdot}{}^{ij}R_{pq}^{\cdot\cdot}{}^{lm}\epsilon_{ijlm}\sqrt{\Delta} = 2R_{pqij}R_{pq}^{\cdot\cdot}{}^{ij} + 4(f_{pq}-t_{pq})^2\,, \qquad (8.4)$$

$$-R_{pq}^{\cdot\cdot}{}^{ij}{}_{,r..}R_{pq}^{\cdot\cdot}{}^{lm}{}_{,r..}\epsilon_{ijlm}\sqrt{\Delta} = 2R_{pqij,r...}R_{pq}^{\cdot\cdot}{}^{ij}{}_{,r..} + 4(f_{pq,r...}-t_{pq,r...})^2\,, \qquad (8.5)$$

and

$$-\sqrt{\Delta}\,\epsilon_{abcd}\{2R_{st}^{\cdot\cdot}{}^{l[a}R_{pql,r...}^{\cdot\cdot\cdot\,b]} - i(f_{st}-t_{st})R_{pq}^{\cdot\cdot}{}^{[ab]}{}_{,r...} - i(f_{pq,r...}-t_{pq,r...})R_{st}^{\cdot\cdot}{}^{[ba]}\}$$
$$= 2\{2R_{st\cdot[c}^{\cdot\cdot l}R_{|pql|d],r...} - i(f_{st}-t_{st})R_{pq[cd],r...} - i(f_{pq,r...}-t_{pq,r...})R_{st[cd]}\}\,. \qquad (8.6)$$

So we have the following result: *The condition of integrability of the fundamental equation for* ψ *is either* [(8.1), (8.2), (8.3)] *or* [(8.4), (8.5), (8.6)].

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(1) See **Note 9**, p. 187.

§ 9. Further consideration of the condition (4.4).

In the section 4, when we found the condition that the right hand side of (4.4) is independent of l, we considered that the coefficient of ψ in the right side of (4.4) is independent of l for all values of ψ; in this section we will consider certain cases where this is not so.

In the multiplier $(I+\Lambda_m\delta x^m)$ of $ds\psi$ which appears by parallel displacement, we can arbitrarily modify the term which vanishes when it is multiplied to $ds\psi$. Therefore when we take Λ_m+P_m (where $P_m ds\psi=0$) instead of Λ_m in the multiplier, it gives the same equation of parallel displacement. Thus, while we are concerned with different parallelisms we may leave the term P_m out of consideration. To express this we write

$$\bar{\Lambda}_m \equiv \Lambda_m \qquad (\mathrm{mod}\ \ ds\psi).$$

when $\bar{\Lambda}_m$ and Λ_m are equivalent except for the term corresponding P_m.

Now in order to obtain the condition that (4.4) is independent of l without taking away ψ from the equation, we introduce the following important theorem: *In the parallel displacement which makes $ds\psi=0$ invariant, in order that*

$$\gamma_l \frac{\partial\psi}{\partial x^m} = \left(\Gamma^i_{lm}\gamma_i - \Lambda_m\gamma_l - \frac{\partial\gamma_l}{\partial x^m}\right)\psi, \tag{9.1}$$

and

$$\frac{\partial\psi}{\partial x^m} = (-\Lambda_m + 2T^{\lambda}_{m}\gamma_\lambda + R_m I)\psi \tag{9.2}$$ [1]

are equivalent when

$$\Gamma^j_{lm} = \left\{ {j \atop lm} \right\} + 4(L^{jk}_m + C^{jk}_m)g_{kl} + \delta^j_l R_m\,,$$ [2]

it is necessary and sufficient that

$$'\Lambda_m \equiv \Lambda_m \qquad (\mathrm{mod}\ \ 'ds\psi), \tag{9.3}$$

when $\Lambda_m \to {'\Lambda_m}$ and $ds \to {'ds}$ by any constant gauge transformation.

Proof. We choose γ_5 so that $g_{5\lambda}=\delta_{5\lambda}$.

First we will obtain the necessary and sufficient condition that

$$'\Lambda_m \equiv \Lambda_m \qquad (\mathrm{mod}\ \ 'ds\psi),$$

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(1) This is the same as (4.12).

(2) cf. (4.9).

by any constant gauge transformation

$$\left.\begin{aligned} W &= I\cosh w + \gamma_5 \sinh w \qquad &(\text{where } \ w = \text{constant})\,, \\ '\gamma_i &= W\gamma_i = V\gamma_i V^{-1} \qquad &(\text{where } \ V^2 = W) \quad . \end{aligned}\right\} \quad (9.4)$$

By the gauge transformation (9.4), (9.1) becomes[1]

$$'\gamma_l \frac{\partial \Psi}{\partial x^m} = \left(\Gamma^i_{lm}{}'\gamma_i - {}'\Lambda_m{}'\gamma_l - \frac{\partial' \gamma_l}{\partial x^m}\right)\Psi\,. \qquad (9.5)$$

Substituting (9.4) into the above

$$\gamma_l \frac{\partial \Psi}{\partial x^m} = \left(\Gamma^i_{lm}\gamma_i - W^{-1}{}'\Lambda_m W\gamma_l - W^{-1}\frac{\partial W}{\partial x^m}\gamma_l - \frac{\partial \gamma_l}{\partial x^m}\right)\Psi\,. \quad (9.6)$$

By the assumption (9.3) we have

$$('\Lambda_m - \Lambda_m) W\gamma_l \Psi = 0\,.$$

From this and (9.1), (9.6) becomes

$$\left(\Lambda_m - W^{-1}\Lambda_m W - W^{-1}\frac{\partial W}{\partial x^m}\right)\gamma_l \Psi = 0\,, \qquad (9.7)$$

but after some calculation[2]

$$\left(\Lambda_m - W^{-1}\Lambda_m W - W^{-1}\frac{\partial W}{\partial x^m}\right) = (-2T^{5i}_m \cosh 2w + T^i_m \sinh 2w)\gamma_5\gamma_i$$
$$+ (2T^{5i}_m \sinh 2w - T^i_m \cosh 2w)\gamma_j\,,$$

therefore (9.7) is written in the form

$$\{(-2T^{5i}_m \cosh 2w + T^i_m \sinh 2w)\gamma_5\gamma_i + (2T^{5i}_m \sinh 2w - T^i_m \cosh 2w)\gamma_i\}\gamma_l \Psi = 0\,.$$

Since the above equation must hold for all w we have

$$(2T^{5i}_m\gamma_5\gamma_i + T^i_m\gamma_i)\gamma_l\Psi = 0\,,$$

from which

$$\{4T^{5i}_m\gamma_5 g_{il} + 2T^i_m g_{il} + \gamma_l(2T^{5i}_m\gamma_5\gamma_i - T^i_m\gamma_i)\}\Psi = 0\,. \qquad (9.8)$$

Multiplying the above γ^l and contracting it by l,

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(1) By the invariancy of parallelism for all transformations.

(2) See **Note 4**, p. 182.

$$(2T^{5i}_{m}\gamma_5\gamma_i - T^{i}_{m}\gamma_i)\Psi = 0$$

and then substituting the above into (9.8) we have

$$(2T^{5i}_{m}\gamma_5 + T^{i}_{m}I)\Psi = 0 . \qquad (9.9)$$

Conversely, reversing the above calculation we have (9.7) from (9.9) and by comparing it with (9.6) we have

$$('\Lambda_m - \Lambda_m) W\gamma_l \Psi = 0 ,$$

so *we know that* (9.9) *is the condition in order that*

$$'\Lambda_m \equiv \Lambda_m \qquad (\mathrm{mod}\ \ 'ds\Psi) ,$$

for all constant gauge transformations.

Next we shall proceed to prove our theorem by using this result. From (9.2) we have

$$\frac{\partial\Psi}{\partial x^m} = (-\Lambda_m + 2T^{\lambda}_{m}\gamma_\lambda + R_m I)\Psi - \frac{1}{g_{ll}}\gamma_l(4T^{5i}_{m}\gamma_5 + 2T^{i}_{m}I)g_{il}\Psi , \quad (9.10)$$

(here we do not sum by l),

therefore the condition that (9.1) and (9.2) are equivalent is that

$$\gamma_l(2T^{5i}_{m}\gamma_5 + 2T^{i}_{m}I)g_{il}\Psi = 0 \qquad \text{(here we do not sum by } l).$$

But the above and (9.9) are equivalent, therefore the condition for equivalency of (9.1) and (9.2) is (9.9) or (9.3).

Further we can prove the following theorem.
If by any constant gauge transformations

$$'\Lambda_m \equiv \Lambda_m \qquad (\mathrm{mod}\ \ 'ds\Psi) ,$$

and the x space has Weyl's connection, the fundamental equation of Ψ *necessarily becomes*

$$\frac{\partial\Psi}{\partial x^m} = \left\{\Gamma_m + \Gamma^5_m\gamma_5 + \left(\frac{1}{2}Q_m - L_m\right)I\right\}\Psi .$$

Proof. In (9.2) if we put

$$\Gamma^i_{jk} = \{^{\,i}_{jk}\} + \frac{1}{2}(Q_j\delta^i_k + Q_k\delta^i_j - Q^i g_{jk}),$$

and multiplying it by γ^l and contracting it by l

$$\frac{\partial\psi}{\partial x^m} = \left\{\Gamma_m + T^5_m\gamma_5 + \left(\frac{1}{2}Q_m - L_m\right)I\right\}\psi - \{T^{5i}_m\gamma_5 + T^i_m\}\gamma_i\psi, \quad (9.11)$$

but from the assumption

$$'\Lambda_m \equiv \Lambda_m \qquad (\text{mod } 'ds\psi),$$

there holds (9.9) and (9.11) becomes

$$\frac{\partial\psi}{\partial x^m} = \left\{\Gamma_m + T^5_m\gamma_5 + \left(\frac{1}{2}Q_m - L_m\right)I\right\}\psi.\text{[1]}$$

§ 10 Notes.

Note 1.

First, if we calculate $\gamma_l\Lambda_m\gamma_l$, i.e.

$$\gamma_l(L^{\lambda\mu}_m\gamma_\lambda\gamma_\mu + L^\lambda_m\gamma_\lambda + L_m I)\gamma_l,$$

we have

$$\gamma_l L^{\lambda\mu}_m\gamma_\lambda\gamma_\mu\gamma_l = \gamma_l\Gamma^{\lambda\mu}_m\gamma_\lambda(2g_{\mu l} - \gamma_l\gamma_\mu)$$

$$= 4\gamma_l L^{\lambda\mu}_m\gamma_\lambda g_{l\mu} + g_{ll}L^{\lambda\mu}_m\gamma_\lambda\gamma_\mu,$$

and

$$\gamma_l L^\lambda_m\gamma_\lambda\gamma_l = \gamma_l L^\lambda_m(2g_{\lambda l} - \gamma_l\gamma_\lambda)$$

$$= 2\gamma_l L^\lambda_m g_{\lambda l} - g_{ll}L^\lambda_m\gamma_\lambda,$$

$$\therefore\quad \gamma_l\Lambda_m\gamma_l = g_{ll}(L^{\lambda\mu}_m\gamma_\lambda\gamma_\mu - L^\lambda_m\gamma_\lambda) + (4L^{\lambda\mu}_m\gamma_l\gamma_\lambda + 2L^\mu_m\gamma_l)g_{\mu l} + g_{ll}L_m I. \quad (\text{N1.1})$$

Similarly

$$\Gamma_m\gamma_l - \gamma_l\Gamma_m = (C^{\lambda\mu}_m\gamma_\lambda\gamma_\mu + C^\lambda_m\gamma_\lambda + C_m I)\gamma_l - \gamma_l(C^{\lambda\mu}_m\gamma_\lambda\gamma_\mu + C^\lambda_m\gamma_\lambda + C_m I)$$

$$= 4C^{\lambda\mu}_m g_{\mu l}\gamma_\lambda + 2C^\sigma_m\gamma_{[\sigma}\gamma_{l]}.$$

---

(1) This equation is identical with (5.7). Hence we have the following result: that (4.4) is independent of the suffix l for all values of Ψ is equivalent to that (9.3) holds for all constant gauge transformations.

$$\therefore \quad \gamma_l \frac{\partial \gamma_l}{\partial x^m} = \gamma_l[\{{}^{\sigma}_{lm}\}\gamma_\sigma + 4C^{\lambda\mu}_m g_{\mu l}\gamma_\lambda + 2C^\sigma_m \gamma_{[\sigma}\gamma_{l]}]$$

$$= \{{}^{\sigma}_{lm}\}\gamma_l\gamma_\sigma + 4C^{\lambda\mu}_m g_{\mu l}\gamma_l\gamma_\lambda - 2g_{ll}C^\sigma_m\gamma_\sigma + 2C^\sigma_m g_{\sigma l}\gamma_l,$$

(not summing by l). (N1.2)

Substituting (N1.1) and (N1.2) into (4.3), we have

$$\frac{\partial \Psi}{\partial x^m} = \frac{1}{g_{ll}}[\Gamma^j_{lm}\gamma_l\gamma_j - g_{ll}(L^{\lambda\mu}_m\gamma_\lambda\gamma_\mu - L^\lambda_m\gamma_\lambda + L_m I) + 2g_{ll}C^\sigma_m\gamma_\sigma$$

$$-(4L^{\lambda\mu}_m\gamma_l\gamma_\lambda + 2L^\mu_m g_l)g_{\mu l} - \{{}^{\sigma}_{lm}\}\gamma_l\gamma_\sigma - 4C^{\lambda\mu}_m\gamma_l\gamma_\lambda g_{\mu l}$$

$$-2C^\sigma_m g_{\sigma l}\gamma_l]\Psi. \tag{N1.3}$$

Rewriting above the equation by putting n for l, equating its right hand side to that of (N1.3) and comparing the coefficients of the bases of sedenion of the resulting equation, we have the following equations:

$(L^\lambda_m + C^\lambda_m)g_{\lambda l} = 0$ .... (from the coefficient of γ_l), (N1.4)

$\Gamma^j_{lm} = \{{}^{j}_{lm}\} + 4(L^{j\mu}_m + C^{j\mu}_m)g_{\mu l}$ .... (from the coefficients of $\gamma_{[l}\gamma_{5]}$, $j \neq l$), (N1.5)

$0 = \{{}^{5}_{lm}\} + 4(L^{5\mu}_m + C^{5\mu}_m)g_{\mu l}$ .... (from the coefficients of $\gamma_{[l}\gamma_{5]}$), (N1.6)

$\Gamma^l_{lm} - \{{}^{l}_{lm}\} - 4(L^{l\mu}_m + C^{l\mu}_m)g_{\mu l} = \Gamma^n_{nm} - \{{}^{n}_{nm}\} - 4(L^{n\mu}_m + C^{n\mu}_m)g_{\mu n}$

(from the coefficients of I), (N1.7)

where we do not sum by l and n. Since (N1.7) holds for any l and n, we can put the value R_m, which may be any covariant vector.

Further, combining (N1.5) and (N1.7) in one equation, we have

$$\Gamma^j_{lm} = \{{}^{j}_{lm}\} + 4(L^{j\mu}_m + C^{j\mu}_m)g_{\mu l} + \delta^j_l R_m. \tag{N1.8}$$

So we have (N1.4), (N1.6), (N1.8) as the condition that the right hand side of (4.4) is independent of l. And substituting them into (N1.3) we have

$$\frac{\partial \Psi}{\partial x^m} = \{-L^{\lambda\mu}_m\gamma_\lambda\gamma_\mu + L^\lambda_m\gamma_\lambda - L_m I + R_m I + 2C^\sigma_m\gamma_\sigma\}\Psi$$

$$= (-L^{\lambda\mu}_m\gamma_\lambda\gamma_\mu - L^\lambda_m\gamma_\lambda - L_m I)\Psi + \{2(C^\sigma_m + L^\sigma_m)\gamma_\sigma + R_m I)\Psi. \tag{N1.9}$$

Note 2.

Since the parallelism of the vectors in the x space is invariant by all coordinate transformations, the expressions $ds\psi$, $\overline{ds}\psi$, and $\Lambda_m \delta x^m$ in the equation

$$(\overline{ds}\psi)_x = (1+\Lambda_m \delta x^m)(ds\psi)_{x+\delta x}$$

must be invariant by C-transformation. Hence Λ_m is transformed covariantly by C-transformation.

And since, as we see from (4.5), Γ_m is also transformed covariantly, $T^{\lambda}_{m}\gamma_\lambda$ behaves as a covariant vector; and as we have seen in (N1.8) R_m is a covariant vector. Consequently, by C-transformation the operator ∇_m is transformed covariantly.

Note 3.

The parallelism between vectors $\overline{dx}^i$ and dx^i in x space must be invariant by all ψ-transformation:

$$\left.\begin{aligned} '\psi &= S\psi\,, \\ '\gamma_i &= SW\gamma_i S^{-1}\,. \end{aligned}\right\}.$$

In the equation of parallelism

$$(\gamma_i \overline{dx}^i \psi)_x = (I+\Lambda_m \delta x^m)(\gamma_i dx^i \psi)_{x+\delta x}\,, \tag{N3.1}$$

if we apply the ψ-transformation and find the condition of invariancy of the correspondence between two parallel vectors $\overline{dx}^i$ and dx^i, we have

$$('\gamma_i \overline{dx}^{i\prime} \psi)_x = (I+'\Lambda_m \delta x^m)('\gamma_i dx^i \psi)_{x+\delta x}\,,$$

or

$$(SW\gamma_i \overline{dx}^i \psi)_x = (I+'\Lambda_m \delta x^m)(SW\gamma_i dx^i \psi)_{x+\delta x}\,,$$

$$\therefore \quad (\gamma_i \overline{dx}^i \psi)_x = (SW)_x^{-1}(I+'\Lambda_m \delta x^m)(SW\gamma_i dx^i \psi)_{x+\delta x}\,, \tag{N3.2}$$

and from (N3.1) and (N3.2) we have

$$\{(I+\Lambda_m \delta x^m) - (SW)_x^{-1}(I+'\Lambda_m \delta x^m)(SW)_{x+\delta x}\}(ds\psi)_{x+\delta x} = .$$

Neglecting terms higher than the second order of δx^i, we have

$$\left\{\Lambda_m + \frac{\partial (SW)^{-1}}{\partial x^m} - (SW)^{-1}{}'\Lambda_m SW\right\}_{x+\delta x}(ds\psi)_{x+\delta x} = 0\,,$$

$$\therefore \qquad 'A_m = SWA_mW^{-1}S^{-1}+SW\frac{\partial(SW)^{-1}}{\partial x^m}.$$

From this we know that by S-transformation: $'\psi = S\psi$, A_m undergoes the transformation

$$'A_m = SA_mS^{-1}+S\frac{\partial S^{-1}}{\partial x^m}, \tag{N3.3}$$

and by G-transformation: $'\gamma_i = W\gamma_i$, the same undergoes the transformation

$$'A_m = WA_mW^{-1}+W\frac{\partial W^{-1}}{\partial x^m}. \tag{N3.4}$$

Next we will consider the transformation of T^λ_m by S-transformation. By S-transformation:

$$\left.\begin{aligned} '\psi &= S\psi \\ '\gamma_i &= S\gamma_iS^{-1}, \end{aligned}\right\}$$

$$\frac{\partial\gamma_i}{\partial x^m} = \left\{{\lambda \atop im}\right\}\gamma_\lambda+\Gamma_m\gamma_i-\gamma_i\Gamma_m \tag{N3.5}$$

becomes

$$\frac{\partial'\gamma_i}{\partial x^m} = \left\{{\lambda \atop im}\right\}'\gamma_\lambda+'\Gamma_m{}'\gamma_i-'\gamma_i{}'\Gamma_m,$$

or

$$\frac{\partial S}{\partial x^m}S^{-1}{}'\gamma_i+S\frac{\partial\gamma_i}{\partial x^m}S^{-1}+'\gamma_iS\frac{\partial S^{-1}}{\partial x^m} = \left\{{\lambda \atop im}\right\}'\gamma_\lambda+'\Gamma_m{}'\gamma_i-'\gamma_i{}'\Gamma_m.$$

Substituting (N3.5) into the above,

$$\frac{\partial S}{\partial x^m}S^{-1}{}'\gamma_i+'\gamma_iS\frac{\partial S^{-1}}{\partial x^m} = '\Gamma_m{}'\gamma_i-'\gamma_i{}'\Gamma_m-S\Gamma_mS^{-1}{}'\gamma_i+'\gamma_iS\Gamma_mS^{-1},$$

$$\therefore \quad \left(\frac{\partial S}{\partial x^m}S^{-1}-'\Gamma_m+S\Gamma_mS^{-1}\right)'\gamma_i = '\gamma_i\left(\frac{\partial S}{\partial x^m}S^{-1}-'\Gamma_m+S\Gamma_mS^{-1}\right),$$

$$\therefore \qquad '\Gamma_m = S\Gamma_mS^{-1}-S\frac{\partial S^{-1}}{\partial x^m}. \tag{N3.6}$$

From (N3.3) and (N3.6)

$$'(\Lambda_m+\Gamma_m) = S(\Lambda_m+\Gamma_m)S^{-1},$$

$$\therefore \qquad 'T^{\lambda}_{m} = T^{\lambda}_{m}.$$

This is the required transformation of T^{λ}_{m} by S-transformation.

Note 4.

By G-transformation:

$$'\gamma_i = W\gamma_i, \qquad \text{where} \qquad W = I\cosh w + A^{\lambda}\gamma_{\lambda}\sinh w,$$

from (N3.4) we have

$$'\Lambda_m = W\Lambda_m W^{-1} + W\frac{\partial W^{-1}}{\partial x^m}. \tag{N4.1}$$

But from the property of W, we can easily see that

$$'\gamma_i = W\gamma_i = V\gamma_i V^{-1},$$

where

$$V = W^{\frac{1}{2}} = I\cosh\frac{W}{2} + A^{\lambda}\gamma_{\lambda}\sinh\frac{W}{2}.$$

Therefore we have the expression for transformation of Γ_m, similarly as we have obtained (N4.1),

$$'\Gamma_m = V\Gamma_m V^{-1} - V\frac{\partial V^{-1}}{\partial x^m}. \tag{N4.2}$$

Now in order to reduce (N4.1) and (N4.2) in more concrete forms, we consider the following transformation of $\bar{\gamma}_{\lambda}$'s:

$$\left.\begin{aligned} \bar{\gamma}_i &= \gamma_i \\ \bar{\gamma}_5 &= A^{\lambda}\gamma_{\lambda} \end{aligned}\right\}, \tag{N4.3}$$

where

$$A^{\lambda}g_{\lambda i} = 0, \qquad A^{\lambda}A^{\mu}g_{\lambda\mu} = 1,$$

by which we easily see that

$$\bar{g}_{ij} = g_{ij}\,, \qquad \bar{g}_{5\lambda} = \delta_{5\lambda}\,.$$

Then the expression for W becomes

$$W = I\cosh w + \bar{\gamma}_5 \sinh w\,.$$

Expanding Λ_m in sedenion of the base $\bar{\gamma}_\lambda$ and taking account of the relation:

$$W^{-1} = I\cosh w - \bar{\gamma}_5 \sinh w\,,$$

the first term of the right hand side of (N4.1) becomes

$$\begin{aligned} W\Lambda_m W^{-1} &= \bar{L}^{ij}_m\bar{\gamma}_i\bar{\gamma}_j + \bar{L}^5_m\bar{\gamma}_5 + \bar{L}_m I + W\bar{L}^{5i}_m(\bar{\gamma}_5\bar{\gamma}_i - \bar{\gamma}_i\bar{\gamma}_5)W^{-1} + W\bar{L}^i_m\bar{\gamma}_i W^{-1} \\ &= \bar{L}^{ij}_m\bar{\gamma}_i\bar{\gamma}_j + \bar{L}^5_m\bar{\gamma}_5 + W^2\{\bar{L}^{5i}_m(\bar{\gamma}_5\bar{\gamma}_i - \bar{\gamma}_i\bar{\gamma}_5) + \bar{L}^i_m\bar{\gamma}_i\} \\ &= \bar{L}^{ij}_m\bar{\gamma}_i\bar{\gamma}_j + \bar{L}^5_m\bar{\gamma}_5 + (2\cosh 2w\bar{L}^{5i}_m + \sinh 2wL^i_m)\bar{\gamma}_5\bar{\gamma}_i \\ &\qquad + (2\sinh 2w\bar{L}^{5i}_m + \cosh 2w\bar{L}^i_m)\bar{\gamma}_i\,, \end{aligned} \tag{N4.4}$$

and the second term:

$$\begin{aligned} W\frac{\partial W^{-1}}{\partial x^m} &= -\bar{\gamma}_5\frac{\partial w}{\partial x^m} - \frac{1}{2}\{\sinh 2w + (\cosh 2w - 1)\bar{\gamma}_5\} \\ &\qquad \{4\bar{C}^{\lambda 5}_m\bar{\gamma}_\lambda + \bar{C}^\lambda_m(\bar{\gamma}_\lambda\bar{\gamma}_5 - \bar{\gamma}_5\bar{\gamma}_\lambda)\} \\ &= -\bar{\gamma}_5\frac{\partial w}{\partial x^m} + \frac{1}{2}\{4(\cosh 2w - 1)\bar{C}^{5i}_m + 2\sinh 2w\bar{C}^i_m\}\bar{\gamma}_5\bar{\gamma}_i \\ &\qquad + \frac{1}{2}\{4\sinh 2w\bar{C}^{5i}_m + 2(\cosh 2w - 1)\bar{C}^i_m\}\bar{\gamma}_i\,; \end{aligned} \tag{N4.5}$$

therefore, (N4.1) becomes

$$\begin{aligned} {}'\Lambda_m = W\Lambda_m W^{-1} + W\frac{\partial W^{-1}}{\partial x^m} &= \bar{L}^{\lambda\mu}_m\bar{\gamma}_\lambda\bar{\gamma}_\mu + \bar{L}^\lambda_m\bar{\gamma}_\lambda + \bar{L}_m I \\ &\quad + \{2(\cosh 2w - 1)\bar{T}^{5i}_m + \sinh\bar{T}^i_m\}\bar{\gamma}_5\bar{\gamma}_i \\ &\quad + \{2\sinh 2w\bar{T}^{5i}_m + (\cosh 2w - 1)\bar{T}^i_m\}\bar{\gamma}_i - \frac{\partial w}{\partial x^m}\bar{\gamma}_5\,, \end{aligned}$$

where

$$\overline{T}^{5i}_{m} = \overline{L}^{5i}_{m} + \overline{C}^{5i}_{m}, \qquad \overline{T}^{i}_{m} = \overline{L}^{i}_{m} + \overline{C}^{i}_{m}.$$

And from (4.8), (4.10), we have

$$\overline{T}^{5i}_{m} = 0, \qquad \overline{T}^{i}_{m} = 0,$$

so we have

$$'\Lambda_m = \Lambda_m - \frac{\partial w}{\partial x^m}\bar{\gamma}_5, \qquad '\overline{L}^{5}_{m} = \overline{L}^{5}_{m} - \frac{\partial w}{\partial x^m}\bar{\gamma}_5. \qquad \text{(N4.6)}$$

By a process exactly similar to the above, (N4.2) can be calculated as follows

$$'\Gamma_m = \Gamma_m + \frac{1}{2}\frac{\partial w}{\partial x^m}\bar{\gamma}_5, \qquad '\overline{C}^{5}_{m} = \overline{C}^{5}_{m} + \frac{1}{2}\frac{\partial w}{\partial x^m}\bar{\gamma}_5, \qquad \text{(N4.7)}$$

and from (N4.6), (N4.7)

$$'(\overline{T}^{5}_{m}\bar{\gamma}_5) = \overline{T}^{5}\bar{\gamma}_5 - \frac{\partial w}{\partial x^m}\bar{\gamma}_5 \qquad (\text{from } V\bar{\gamma}_5V^{-1} = \bar{\gamma}_5).$$

Consequently we have

$$'\nabla_m = \frac{\partial}{\partial x^m} + '\Lambda_m - 2'\overline{T}^{5}_{m}{}'\bar{\gamma}_5$$

$$= \frac{\partial}{\partial x^m} + \Lambda_m - 2\overline{T}^{5}_{m}\bar{\gamma}_5 = \nabla_m.$$

If we transform the above result back by using (N4.3), we have

$$T^{\lambda}_{m} = \overline{T}^{5}_{m}A^{\lambda}$$

and therefore

$$'\nabla_m = \frac{\partial}{\partial x^m} + '\Lambda_m - 2'T^{\lambda}_{m}{}'\gamma_\lambda$$

$$= \frac{\partial}{\partial x^m} + \Lambda_m - 2T^{\lambda}_{m}\gamma_\lambda = \nabla_m.$$

Note 5.

Taking $\bar{\gamma}_\lambda$'s as in Note **4,** (N1.8) becomes

$$\overline{R}_m g^{jk} = (\Gamma^{(i}_{lm} - \{^{(j}_{lm}\})g^{k)l}\,,$$

of which the right side is independent of the ψ-transformation, therefore

$$'\overline{R}_m = \overline{R}_m\,. \tag{N5.1}$$

But since the equation $\nabla_m\psi = R_m\psi$ is independent of the choice of the base of sedenion, (N5.1) holds by transforming the bases back in general case, namely

$$'R_m = R_m\,.$$

Note 6.

(7.2) is rewritten as follows

$$\frac{\partial\psi}{\partial x^m} = (\varDelta_m - \varTheta_m + T^5_{m}\gamma_5)\psi\,, \tag{N6.1}$$

where

$$\varDelta_m = \Gamma_m + (R_m - L_m)I\,,$$

$$\varTheta_m = \frac{1}{4}(\Gamma^i_{lm} - \{^{\,i}_{lm}\})g^{kl}\gamma_{[i}\gamma_{k]} = \frac{1}{4}M^{k\cdot i}_{\cdot m}\gamma_{[i}\gamma_{k]}\,,$$

$$\therefore\quad 2\frac{\partial^2\psi}{\partial x^{[m}\partial x^{l]}} = \left\{\frac{\partial\varDelta_l}{\partial x^m} - \frac{\partial\varDelta_m}{\partial x^l} + (\varDelta_l - \varTheta_l + T^5_l\gamma_5)(\varDelta_m - \varTheta_m + T^5_m\gamma_5)\right.$$

$$-(\varDelta_m - \varTheta_m + T^5_m\gamma_5)(\varDelta_l - \varTheta_l + T^5_l\gamma_5)$$

$$\left. + \frac{\partial}{\partial x^m}(-\varTheta_l + T^5_l\gamma_5) - \frac{\partial}{\partial x^l}(-\varTheta_m + T^5_m\gamma_5)\right\}\psi,$$

therefore as the condition of integrability of (N6.1) we have

$$\left[\left(\frac{\partial\Gamma_l}{\partial x^m} - \frac{\partial\Gamma_m}{\partial x^l} + \Gamma_l\Gamma_m - \Gamma_m\Gamma_l\right) - \left(\frac{\partial\varTheta_l}{\partial x^m} - \frac{\partial\varTheta_m}{\partial x^l} + \Gamma_l\varTheta_m - \varTheta_m\Gamma_l\right.\right.$$

$$\left. - \Gamma_m\varTheta_l + \varTheta_l\Gamma_m\right) + (\varTheta_l\varTheta_m - \varTheta_m\varTheta_l)$$

$$+ \left(\frac{\partial}{\partial x^m}(R_l - L_l) - \frac{\partial}{\partial x^l}(R_m - L_m)\right)I + \left(\frac{\partial T^5_l\gamma_5}{\partial x^m} - \frac{\partial T^5_m\gamma_5}{\partial x^l}\right)$$

$$\left. + (\Gamma_l T^5_m\gamma_5 - \Gamma_m T^5_l\gamma_5 - T^5_m\gamma_5\Gamma_l + T^5_l\gamma_5\Gamma_m)\right]\psi = 0\,. \tag{N6.2}$$

The first term in the above equation:

$$\frac{1}{4}\cdot K_{lm}{}^{ij}\gamma_i\gamma_j\,,$$

the second term:

$$-\frac{1}{4}\{\mathring{\nabla}_m M_l^{j\cdot i}-\mathring{\nabla}_l M_m^{j\cdot i}\}\gamma_{[i}\gamma_{j]}\,,$$

where $\mathring{\nabla}_m T^{\cdots}_{\cdots}$ expresses the covariant derivative of $T^{\cdots}_{\cdots}$ with respect to $\{^{\,i}_{jk}\}$, the third term:

$$M^{[q\cdot p]}_{\cdot l}\gamma_p(2g_{qi}-\gamma_i\gamma_q)M^{i\cdot j}_{\cdot m}\gamma_j-\theta_m\theta_l$$

$$=(4M^{[q\cdot p]}_{\cdot l}\gamma_p g_{qi}+\gamma_i M^{[q\cdot p]}_{\cdot l}\gamma_p\gamma_q)\gamma_j M^{i\cdot j}_{\ m}-\theta_m\theta_l$$

$$=(4M^{[q\cdot p]}_{\cdot l}\gamma_p g_{qi}\gamma_j+4\gamma_i M^{[q\cdot p]}_{\cdot l}\gamma_p g_{qj})M^{[i\cdot j]}_{\cdot m}$$

$$=4(M^{[q\cdot p]}_{\cdot l}M^{[i\cdot j]}_{\cdot m}-M^{[q\cdot p]}_{\ m}M^{[i\cdot j]}_{\cdot l})g_{pi}\gamma_q\gamma_j\,,$$

the fifth term;

$$\left(\frac{\partial T^5_l}{\partial x^m}-\frac{\partial T^5_m}{\partial x^l}\right)\gamma_5+2T^5_{[l}\{4C^{i5}_{m]}\gamma_{|i|}+C^i_{m]}(\gamma_i\gamma_5-\gamma_5\gamma_i)\}\,,$$

the sixth term:

$$2T^5_{[m}\{4C^{i5}_{l]}\gamma_{|i|}+C^i_{l]}(\gamma_i\gamma_5-\gamma_5\gamma_i)\}\,.$$

Consequently, (N6.3) can be written in

$$\left[\frac{1}{4}R_{lm}{}^{ij}\gamma_{[i}\gamma_{j]}+\left(\frac{\partial T^5_l}{\partial x^m}-\frac{\partial T^5_m}{\partial x^l}\right)\gamma_5+\left\{\frac{\partial}{\partial x^m}(R_l-L_l)\right.\right.$$

$$\left.\left.-\frac{\partial}{\partial x^l}(R_m-L_m)\right\}I\right]\psi=0\,, \qquad \text{(N6.3)}$$

where R_{lmij} is the curvature tensor made from Γ^i_{jk}.

Note 7.

Differentiating the first term of (N6.3) and calculating it by using (5.3), we have

$$\frac{\partial}{\partial x^r}(R_{l\dot{m}}{}^{ij}\gamma_{[i}\gamma_{j]}\Psi)$$

$$= \left[\frac{\partial R_{l\dot{m}}{}^{ij}}{\partial x^r}\gamma_{[i}\gamma_{j]} + R_{l\dot{m}}{}^{[ij]}\{(\Gamma^p_{ir}\gamma_p - R_r\gamma_i - 2T^5_r\gamma_{[5}\gamma_{i]})\gamma_i - (\Lambda_r - 2T^5_r\gamma_5)\gamma_i\gamma_j\right.$$

$$+\gamma_i(\Gamma^p_{jr}\gamma_p - R_r\gamma_j - 2T^5_r\gamma_{[5}\gamma_{j]}) + \gamma_i\gamma_j(\Lambda_r - 2T^5_r\gamma_5)\}$$

$$\left.+ R_{l\dot{m}}{}^{[ij]}\gamma_i\gamma_j(-\Lambda_r + 2T^5_r\gamma_5 + R_r I)\right]\Psi$$

$$= [R_{l\dot{m}}{}^{ij}{}_{,r}\gamma_{[i}\gamma_{j]} + \{-R_{l\dot{m}}{}^{ij}(\Lambda_r - 2T^5_r\gamma_5 - R_r)$$

$$+ R_{p\dot{m}}{}^{ij}(\Gamma^p_{lr} - R_r\delta^p_l) + R_{l\dot{p}}{}^{ij}(\Gamma^p_{mr} - R_r\delta^p_m)\}\gamma_{[i}\gamma_{j]}\Psi\,,$$

and similarly from the second term of (N6.3),

$$\frac{\partial}{\partial x_r}(t_{lm}\gamma_5\Psi) = \left[\left(\frac{\partial}{\partial x_r}t_{lm}\right)\gamma_5 + t_{lm}\{-(\Lambda_r - 2T^5_r\gamma_5)\gamma_5 + \gamma_5(\Lambda - 2T^5_r\gamma_5)\}\right.$$

$$\left.+ t_{lm}\gamma_5(-\Lambda_r + 2T^5_r\gamma_5 + R_r I)\right]\Psi$$

$$= [t_{lm,r}\gamma_5 - t_{lm}(\Lambda_r - 2T^5_r\gamma_5 - R_r I)\gamma_5 + \{t_{pm}(\Gamma^p_{lr} - R_r\delta^p_l)$$

$$+ t_{lp}(\Gamma^p_{mr} - R_r\delta^p_m)\}\gamma_5]\Psi\,,$$

where $F^{\cdot\cdot\cdot}_{\cdot\cdot\cdot,r}$ expresses the covariant derivative of $F^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}$ with respect to $\Gamma^i_{jk} - \delta^i_j R_k$, therefore the equation which is obtained from (N6.3) by differentiating with respect to x^r is written as follows by again using the relation (N6.3):

$$\left(\frac{1}{4}R_{l\dot{m}}{}^{ij}{}_{,r}\gamma_{[i}\gamma_{j]} + t_{lm,r}\gamma_5 + f_{lm,r}I\right)\Psi = 0\,.$$

Note 8.

Multiplying (7.5) by $((a\;b\;c\;d))\,h^c_{[s}h^d_{t]}$ and taking the summation by a, b, c, d

$$\frac{1}{2}\epsilon_{pqst}DR_{l\dot{m}}{}^{pq} = \sum_{c,\,d} R_{l\dot{m}}{}^{ii}h^c_{[i}h^d_{j]}h^c_{[s}h^d_{t]}\,, \tag{N8.1}$$

where

$$D = \begin{vmatrix} h^1_1 & . & . & h^1_4 \\ . & . & . & . \\ h^4_1 & . & . & h^4_4 \end{vmatrix},$$

and after a little calculation the right hand side of (N8.1) is rewritten in the form

$$R_{lm}^{\cdot\cdot[ij]} g_{is} g_{jt} \,,$$

and

$$D^2 = \begin{vmatrix} g_{11} & \cdot & \cdot & g_{14} \\ \cdot & \cdot & \cdot & \cdot \\ g_{41} & \cdot & \cdot & g_{44} \end{vmatrix} = \Delta \,,$$

so (N8.1) becomes

$$\frac{1}{2} \epsilon_{pqst} \sqrt{\Delta}\, R_{lm}^{\cdot\cdot pq} = R_{lm[st]} \,.$$

Note 9.

The third equation of (6.10) becomes

$$\begin{aligned} & R_{st}^{\cdot\cdot[ij]} R_{pq}^{\cdot\cdot lm}{}_{,r\ldots} (h_i^1 h_j^4 - h_i^2 h_j^3)(h_l^1 h_m^2 i - h_l^1 h_m^3 - h_l^2 h_m^4 - h_l^3 h_m^4 i) \\ & \quad + (f_{st} + t_{st}) R_{pq}^{\cdot\cdot lm}{}_{,r\ldots} (h_l^1 h_m^2 i - h_l^1 h_m^3 - h_l^2 h_m^4 - h_l^3 h_m^4 i) \\ = {} & R_{st}^{\cdot\cdot[ij]} R_{pq}^{\cdot\cdot[lm]}{}_{,r\ldots} (h_l^1 h_m^4 - h_l^2 h_m^3)(h_i^1 h_j^2 i - h_i^1 h_j^3 - h_i^2 h_j^4 - h_i^3 h_j^4 i) \\ & \quad + (f_{pq,r\ldots} + t_{pq,r\ldots}) R_{st}^{\cdot\cdot[ij]} (h_i^1 h_j^2 i - h_i^1 h_j^3 - h_i^2 h_j^4 - h_i^3 h_j^4) \,, \end{aligned}$$

from which we have the following equation by suitably changing the arrangements of terms and using certain identities(1)

$$\begin{aligned} & 2R_{st}^{\cdot\cdot[ij]} R_{pq}^{\cdot\cdot[lm]}{}_{,r\ldots} \sum_a h_i^a h_l^a (h_{[j}^4 h_{m]}^a i - h_{[j}^4 h_{m]}^3 - h_{[j}^1 h_{m]}^2 - h_{[i}^1 h_{m]}^3 i) \\ & \quad + (f_{st} + t_{st}) R_{pq}^{\cdot\cdot[jm]}{}_{,r\ldots} (h_j^1 h_m^2 i - h_j^1 h_m^3 - h_j^2 h_m^4 - h_j^3 h_m^4 i) \\ & \quad - (f_{pq,r\ldots} + t_{pq,r\ldots}) R_{st}^{\cdot\cdot[jm]} (h_j^1 h_m^2 i - h_j^i h_m^3 - h_j^2 h_m^4 - h_j^3 h_m^4 i) = 0 \,. \end{aligned}$$

But the above equation may be written in the following forms

$$N^{jm} (h_{[j}^4 h_{m]}^2 i - h_{[j}^4 h_{m]}^3 - h_{[j}^1 h_{m]}^2 - h_{[j}^1 h_{m]}^3 i = 0 \,, \qquad \text{(N9.1)}$$

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(1) $(h_{[i}^1 h_{j]}^4 - h_{[i}^2 h_{j]}^3)(i h_{[l}^1 h_{m]}^2 - h_{[l}^1 h_{m]}^3 - h_{[l}^2 h_{m]}^4 - h_{[l}^3 h_{m]}^4$

$$\begin{aligned} &= h_{[i}^1 h_{[l}^1 (h_{j]}^4 h_{m]}^2 - h_{j]}^4 h_{m]}^3) + h_{[i}^2 h_l^2 (h_{j]}^3 h_m^1 i + h_{j]}^3 h_m^4) + h_{[i}^3 h_l^3 (h_{j]}^2 h_m^1 - h_{j]}^2 h_m^{4i}) \\ &\qquad + h_{[i}^4 h_l^4 (-h_{j]}^1 h_m^2 - h_{j]}^1 h_m^3) \\ &= (h_{[i}^1 h_l^1 + h_{[i}^2 h_l^2 + h_{[i}^3 h_l^3 + h_{[i}^4 h_l^4)(h_{j]}^4 h_m^2 i - h_{j]}^4 h_m^3 - h_{j]}^1 h_m^2 - h_{j]}^1 h_m^3 i) \,. \end{aligned}$$

where

$$N^{jm} = 2R_{st}^{\cdot\cdot l[j}R_{pql}^{\cdot\cdot\cdot m]}{}_{,r\dots} - i(f_{st}R_{pq}^{\cdot\cdot[jm]}{}_{,r\dots} + f_{pq,r\dots}R_{st}^{\cdot\cdot[mj]}) .$$

If we multiply (N9.1) by $h_c^{[1}h_d^{4]} - h_c^{[2}h_d^{3]}$,[(1)] we have

$$g_{[c|k|}N^{km}(h_{d]}^2 h_m^4 + h_{d]}^3 h_m^4 i + h_{d]}^1 h_m^3 + h_{d]}^2 h_m^1 i) = 0 \qquad \text{(N9.2)}$$

and if we multiply $\bar{h}_{[i}^c \bar{h}_{j]}^d$ and contract by c, d, we have

$$N^{km}h_k^{[i}(\delta^{j]2}h_m^4 + \delta^{j]3}h_m^4 i + \delta^{j]1}h_m^3 + \delta^{j]2}h_m^1 i) = 0 ,$$

from which we get three equations (by putting $i, j = 1, 2, 3, 4$)

$$\left.\begin{aligned} N^{ij}h_i^2h_j^3 &= N^{ij}h_i^1h_j^4 \\ N^{ij}h_i^2h_j^4 &= N^{ij}h_i^3h_j^1 \\ N^{ij}h_i^3h_j^4 &= N^{ij}h_i^1h_j^2 \end{aligned}\right\} , \qquad \text{(N9.3)}$$

By the same process as used when we obtained (7.6) from (7.5), (N9.3) can be written as follows

$$\sqrt{\Delta}\ \epsilon_{abcd}N^{cd} = 2N_{ab} ,$$

or

$$\sqrt{\Delta}\ \epsilon_{abcd}\{2R_{st}^{\cdot\cdot l[a}R_{pql}^{\cdot\cdot\cdot b]}{}_{,r\dots} - i(f_{st}R_{pq}^{\cdot\cdot[ab]}{}_{,r\dots} + f_{pq,r\dots}R_{st}^{\cdot\cdot[ba]})\}$$

$$= 2\{2R_{st\cdot[c}^{\cdot\cdot l}R_{|pql|d],r\dots} - i(f_{st}R_{pq[cd],r\dots} + f_{pq,r\dots}R_{st[cd]})\} . \qquad \text{(N9.4)}$$

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(1) $N^{jm}\{h_{[c}^1 h_{|j|}^1(-h_{d]}^4 h_m^2 - h_{d]}^4 h_m^3 i) + h_{[c}^2 h_{|j|}^2(h_{d]}^3 h_m^4 - h_{d]}^3 h_m^1)$

$+ h_{[c}^3 h_{|j|}^3(h_{d]}^2 h_m^4 + h_{d]}^2 h_m^1 i) + h_{[c}^4 h_{|j|}^4(h_{d]}^2 h_m^1 i + h_{d]}^3 h_m^1)\}$

$= N^{jm}(h_{[c}^1 h_{|j|}^1 + h_{[c}^2 h_{|j|}^2 + h_{[c}^3 h_{|j|}^3 + h_{[c}^4 h_{|j|}^4)(h_{d]}^2 h_m^4 + h_{d]}^3 h_m^4 i + h_{d]}^1 h_m^3 + h_{d]}^2 h_m^1 i)$.