Application of the Theory of Set Functions to the Mixing of Fluids.

By

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In a treatize on probablity,(1) H. Poincaré proposed, a problem involving the mixing of liquids which may be summarized as follows: Consider a liquid contained in a vessel, and let the liquid be in steady motion. It is assumed that one part of the liquid in the vessel is distinguished from the other by some visible quality, for example it is red whilst the other part is colourless; but both parts obey the same law of motion. At the beginning, the coloured parts may be distributed in the vessel in any manner. But after a certain length of time they are distributed uniformly in the vessel. How can we explain this phenomenon? Several attempts have been made to solve this problem.(2) In this paper, I have solved it by using the properties of set functions, and I consider the general case, i. e. the unsteady motion of compressible fluids. Of course, the following mathematical treatment can also be applied to the diffusion of fluids.

1. In order to interpret the physical phenomena mathematically, we must schematise these phenomena, so that we can completely describe the physical changes by mathematical means. First we construct the schema for the mixing of fluids.

Let a closed vessel be filled with a fluid. We may denote the vessel by a Borel set A in the Euclidean space of three dimensions. Let $\mathfrak{K}$ be the closed family (σ-Körper) of all Borel subsets of A. Denote the quantity of the fluid contained in a Borel set E at time t, by $\sigma_t(E)$. Obviously $\sigma_t(E)$ is a completely additive, non-negative set function defined in $\mathfrak{K}$, whose functional value depends upon t. When the motion of the fluid in A is steady, then $\sigma_t(E)$ is independent of t. And when the fluid is imcompressible, the values of the set function $\sigma_t(E)$ are the

(1) H. Poincaré, *Calcul des probabilités*, 2e éd. (1912), 320.

(2) For example, cf. B Hostinský, "Application du Calcul des Probabilités à la Théorie du Mouvement Brownien", Ann. de l'institut H. Poincaré, **3** (1932), 1–74.

same for all congruent sets, as in the case of the Lebesgue measure. In the general case, $\sigma_t(E)$ signifies the case of unsteady motion of a compressible fluid. Let the total quantity of the fluid in A be independent of time, so that $\sigma_t(A) = \sigma_{t'}(A)$ for any t and t'.

Similarly, the distribution of the coloured part of the liquid at time t is also expressed by a completely additive, non-negative set function $\alpha_t(E)$ defined in $\mathfrak{K}$, which is equal to the quantity of the coloured liquid contained in E at time t. Of course, $\alpha_t(E) \leqq \sigma_t(E)$ for any E, and $\alpha_t(A)$ is a constant value independent of t. Hence, $\alpha_t(E)$ is absolutely continuous with respect to $\sigma_t(E)$, and since $0 \leqq D_{\sigma_t(E)}\, \alpha(a)$[1] $\leqq 1$ almost everywhere (σ_t), it belongs to $\mathfrak{L}_2(\sigma_t)$.[2]

The fluid contained in E at time t will or will not be contained in E' at time t' $(t' > t)$. Denote by $\mathfrak{A}_{t',t}(E', E)$ the quantity of that part of the liquid contained in E at time t, which is also contained in E' at time t'. Of course, $\mathfrak{A}_{t',t}(E', E)$ must be a completely additive, non-negative set function defined in $\mathfrak{K}$ as a function of set E', and as a function set E. And

$$\mathfrak{A}_{t',t}(E', A) = \sigma_{t'}(E') , \qquad \mathfrak{A}_{t',t}(A, E) = \sigma_t(E) . \tag{1}$$

Since
$$\mathfrak{A}_{t',t}(E', E) < \mathfrak{A}_{t',t}(E', A) = \sigma_{t'}(E')$$

for any set E', as a function of set E', $\mathfrak{A}_{t',t}(E', E)$ is absolutely continuous with respect to $\sigma_{t'}(E')$, and $D_{\sigma_{t'}(E')}\, \mathfrak{A}_{t',t}(a', E) \leqq 1$ almost everywhere $(\sigma_{t'})$. Hence $\mathfrak{A}_{t',t}(E', E)$ belongs to $\mathfrak{L}_2(\sigma_{t'})$ as a function of set E'. Similarly, $\mathfrak{A}_{t',t}(E', E)$ belongs to $\mathfrak{L}_2(\sigma_t)$ as a function of set E.

Now $\dfrac{\mathfrak{A}_{t',t}(E'. E)}{\sigma_t(E)}$ denotes the probability that a particle contained in E at t will be contained in E' at t'. But, since

$$D_{\sigma_t(E)}\mathfrak{A}_{t',t}(E', a) = \lim_{n\to\infty} \frac{\mathfrak{A}_{t',t}(E', E_n)}{\sigma_t(E_n)} ,$$

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(1) $D_{\sigma(E)}\alpha(a)$ means the derivative of $\alpha(E)$ with respect to $\sigma(E)$ at point a. Of course, when $\alpha(E)$ is absolutely continuous with respect to $\sigma(E)$, $\alpha(E) = \int_E D_{\sigma(E)}\alpha(a)\, d\sigma(E)$. Cf. S. Saks, *Théorie de l'intégrale*, (1933), 255; and F. Maeda, this journal, **2** (1932), 37.

(2) When a complex valued set function $\alpha(E)$ is absolutely continuous withe respect to $\sigma(E)$, and $\int_A |D_{\sigma(E)}\, \alpha(a)|^2 d\sigma(E)$ is finite, then we may say that $\alpha(E)$ belongs to $\mathfrak{L}_2(\sigma)$. $\mathfrak{L}_2(\sigma)$ is a Hilbert space with inner product $(\alpha, \alpha') = \int_A D_{\sigma(E)}\, \alpha(a)\overline{D_{\sigma(E)}\alpha'(a)}\, d\sigma(E)$. Cf. F. Maeda, this journal, **3** (1933), 3-7.

where $\{E_n\}$ is a sequence of Borel sets which converges to the point a,[1] $D_{\sigma_t(E)}\mathfrak{A}_{t',t}(E',a$ denotes the probability that the particle at a at time t will be contained in E' at time t'. Hence

$$\int_A D_{\sigma_t(E)}\mathfrak{A}_{t',t}(E',a)d\alpha_t(E) \tag{2}$$

is the quantity of the red fluid which will probably be contained in E' at time t'. But, since $\alpha_{t'}(E')$ is the quantity of the red fluid contained in E' at time t', we must assume that $\alpha_t(E)$ and $\alpha_{t'}(E')$ are related according to the following equation:

$$\alpha_{t'}(E') = \int_A D_{\sigma_t(E)}\mathfrak{A}_{t',t}(E',a)D_{\sigma_t(E)}\alpha_t(a)d\sigma_t(E)\,,^{(2)} \tag{3}$$

Similarly, $$\int_A D_{\sigma_{t'}(E')}\mathfrak{A}_{t'',t'}(E'',a')d_{E'}\mathfrak{A}_{t',t}(E',E)^{(3)} \quad (t''>t'>t)$$

is the quantity of the liquid contained in E at time t, which will probably be contained in E'' at time t''. Hence, we must assume that $\mathfrak{A}_{t',t}(E',E)$ has the following property:

$$\mathfrak{A}_{t'',t}(E'',E) = \int_A D_{\sigma_{t'}(E')}\mathfrak{A}_{t'',t'}(E'',a')D_{\sigma_{t'}(E')}\mathfrak{A}_{t',t}(a',E)d\sigma_{t'}(E')\,, \tag{4}$$

where $\quad t''>t'>t.$

Thus, *we have constructed the schema for the mixing of fluid. That is, we have systems of set functions* $\sigma_t(E)$, $\alpha_t(E)$, $\mathfrak{A}_{t',t}(E',E)$ *defined for all* t *and* t' ($t'>t\geqq t_0$[4]), *which satisfy* (1), (3) *and* (4).[5] (3) signifies that $\mathfrak{A}_{t',t}(E',E)$ is a kernel of linear transformation between $\mathfrak{L}_2(\sigma_t)$ and $\mathfrak{L}_2(\sigma_{t'})$; and (4) signifies the composition of these two linear transformations.[6]

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(1) Cf. F. Maeda, this journal, **2** (1932), 37.

(2) If we put $\sigma_t(E)$ instead of $\alpha_t(E)$ in (3), we have $\int_A D_{\sigma_t(E)}\mathfrak{A}(E',a)\,d\sigma_t(E) = \mathfrak{A}_{t',t}(E',A)$, which is equal to $\sigma_{t'}(E')$ by (1). Hence the relation (3) is satisfied for $\sigma_t(E)$ and $\sigma_{t'}(E')$.

(3) $\int_A f(a')\,d_{E'}\mathfrak{A}(E',E)$ means the integration of $f(a')$ by $\mathfrak{A}(E',E)$ as a function of set E'.

(4) t_0 is the initial time.

(5) The condition $\sigma_{t'}(A)=\sigma_t(A)$ follows from (1), for, put A instead of E and E' in (1), then $\sigma_{t'}(A)=\mathfrak{A}_{t',t}(A,A)=\sigma_t(A)$. The condition $\alpha_{t'}(A)=\alpha_t(A)$ follows from (3), for, put A instead of E' in (3), then, since $D_{\sigma_t(E)}\mathfrak{A}_{t',t}(A,a)=D_{\sigma_t(E)}\sigma_t(a)=1$, we have $\alpha_{t'}(A)=\int_A D_{\sigma_t(E)}\alpha_t(a)d\sigma_t(E)=\alpha_t(A)$.

(6) Cf. F. Maeda, this journal, **3** (1933), 249-251.

2. *To interpret the mixing of the fluid, we must prove in this schema that*

$$\lim_{t\to\infty} \frac{a_t(E)}{\sigma_t(E)} = k \tag{5}$$

where k is a constant number independent of E. Put

$$M_t = \sup\left[\frac{a_t(E)}{\sigma_t(E)}\right] \quad \text{for any set } E \quad \text{where} \quad \sigma_t(E) > 0\,,$$

$$m_t = \inf\left[\frac{a_t(E)}{\sigma_t(E)}\right] \qquad \text{,,} \qquad \text{,,}$$

Then
$$m_t \leqq D_{\sigma_t(E)} a_t(a) \leqq M_t \tag{6}$$

almost everywhere (σ_t). Then, since $D\sigma_{t(E)}\, \mathfrak{A}_{t',t}(E', a) \geqq 0$ almost everywhere (σ_t), from (3) we have

$$\begin{aligned} a_{t'}(E') &\leqq M_t \int_A D_{\sigma_t(E)} \mathfrak{A}_{t',t}(E', a) d\sigma_t(E) \\ &= M_t \mathfrak{A}_{t',t}(E', A) \\ &= M_t \sigma_{t'}(E') \qquad \text{by (1).} \end{aligned}$$

Hence $\qquad M_{t'} \leqq M_t \qquad$ when $\quad t' > t$.

Similarly $\qquad m_{t'} \geqq m_t \qquad$ when $\quad t' > t$.

Therefore, to prove (5), it is sufficient to show that

$$\lim_{t\to\infty} (M_t - m_t) = 0\,. \tag{7}$$

In order to prove (7), we make the following assumption: Consider a sequence

$$t_0 \leqq t_1 < t_2 < \ldots\ldots < t_n < \ldots\ldots$$

so that $\lim\limits_{n\to\infty} t_n = \infty$; and use n instead of t_n in the suffixes, for example $\sigma_{t_n}(E) = \sigma_n(E)$. *Assume that there exists a sequence $\{\lambda_n\}$ $(1 > \lambda_n \geqq 0)$, so that*

$$\frac{\mathfrak{A}_{n,n-1}(U, E)}{\sigma_n(U)} \geqq \lambda_{n-1} \frac{\mathfrak{A}_{n,n-1}(V, E)}{\sigma_n(V)} \tag{8}$$

for any set U, V, E where $\sigma_n(U)>0$, $\sigma_n(V)>0$, and $\sum_{n=1}^{\infty}\lambda_n$ diverges. In this conditions we prove that

$$\lim_{n\to\infty}(M_n-m_n)=0$$

as follows :(1)

From (3), we have

$$\frac{a_n(U)}{\sigma_n(U)}=\frac{1}{\sigma_n(U)}\int_A D_{\sigma_{n-1}(E)}\mathfrak{A}_{n,n-1}(U,a)D_{\sigma_{n-1}(E)}a_{n-1}(a)\,d\sigma_{n-1}(E)$$

$$=\int_A\left[\frac{D_{\sigma_{n-1}(E)}\mathfrak{A}_{n,n-1}(U,a)}{\sigma_n(U)}-\lambda_{n-1}\frac{D_{\sigma_{n-1}(E)}\mathfrak{A}_{n,n-1}(V,a)}{\sigma_n(V)}\right]$$

$$\times D_{\sigma_{n-1}(E)}\,a_{n-1}(a)d\sigma_{n-1}(E)$$

$$+\frac{\lambda_{n-1}}{\sigma_n(V)}\int_A D_{\sigma_{n-1}(E)}\mathfrak{A}_{n,n-1}(V,a)D_{\sigma_{n-1}(E)}a_{n-1}(a)d\sigma_{n-1}(E)\,.$$

Since, from (8), the expression [......] in the above integral is non-negative, by (6) and (3) we have

$$\frac{a_n(U)}{\sigma_n(U)}\geqq m_{n-1}\int_A\left[\frac{D_{\sigma_{n-1}(E)}\mathfrak{A}_{n,n-1}(U,a)}{\sigma_n(U)}-\lambda_{n-1}\frac{D_{\sigma_{n-1}(E)}\mathfrak{A}_{n,n-1}(V,a)}{\sigma_n(V)}\right]$$

$$\times d\sigma_{n-1}(E)+\frac{\lambda_{n-1}}{\sigma_n(V)}a_n(V)$$

$$=m_{n-1}[1-\lambda_{n-1}]+\lambda_{n-1}\frac{a_n(V)}{\sigma_n(V)}\qquad\text{by (1).}$$

Hence $$\frac{a_n(V)}{\sigma_n(V)}-\frac{a_n(U)}{\sigma_n(U)}\leqq(1-\lambda_{n-1})\left[\frac{a_n(V)}{\sigma_n(V)}-m_{n-1}\right].$$

Since $\dfrac{a_n(V)}{\sigma_n(V)}\leqq M_n\leqq M_{n-1}$, we have

$$\frac{a_n(V)}{\sigma_n(V)}-\frac{a_n(U)}{\sigma_n(U)}\leqq(1-\lambda_{n-1})(M_{n-1}-m_{n-1})\,.$$

(1) In this proof, I have modified the method of A. Kolmogoroff (Math. Ann. **104** (1931), 424-426).

But as this inequality holds for any U and V, we have

$$M_n - m_n \leqq (1-\lambda_{n-1})(M_{n-1} - m_{n-1}) .$$

By multiplying, we have

$$M_n - m_n \leqq \prod_{\nu=1}^{n-1}(1-\lambda_\nu)(M_1 - m_1) .$$

Since $\sum_{\nu=1}^{\infty}\lambda_\nu$ diverges, $\lim_{n\to\infty}\prod_{\nu=1}^{n-1}(1-\lambda_\nu) = 0$. Consequently, we have

$$\lim_{n\to\infty}(M_n - m_n) = 0 .$$

Thus, in condition (8) we have solved the problem. Of course, we cannot eliminate such a condition as (8'. For, consider a particular case: In the vessel A, a part U is separated by a barrier from the other parts $A-U$, so that the fluid in $A-U$ does not enter U. In this case, since $\mathfrak{A}_{n,n-1}(U, E) = 0$ for any $E \leqq A-U$ and n, the condition (8) does not hold, and mixing of the fluid does not occur in A.