On the Space which admits a given Continuous Transformation Group in the extended sense.

By

Takasi SIBATA.

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We shall consider the space in which parallel directions along every curve are unaltered by all the transformations of a given continuous transformation group. This case we call the space which *admits the given group in the extended sense.*[1]

First, by using the notations of Lie's symbols of infinitesimal transformations, we shall obtain the conditions in which a space admits a group in the extended sense.

Next, we will examine the case when the number of the order of the group admitted by a space in the extended sense, is maximum.

Lastly, in the general space which admits a group in the extended sense, we shall obtain the relations between the covariant derivatives and the transformation-derivatives[2] for a vector field.

1. Let us consider a space V_n,[3] and let the coordinates be $x^1, \ldots, x^n$, and the coefficients of connection be L^i_{jk}. Then we can define the parallelism of vectors in V_n by the following infinitesimal transformations:[4]

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(1) When the parallelism of vectors in a space, not parallel directions, is unaltered by all the transformations of a given group, I say that the space admits the group (not in the extended sense). I have treated this case in this journal **4** (1934), 111–126.

(2) T. Sibata, this journal **4** (1934), 116–117.

(3) In this paper we shall employ certain notations due to L.P. Eisenhart, *Non-Riemannian Geometry* (1927).

(4) T. Sibata, *loc. cit.*, 112.

$$T_i = \frac{\partial}{\partial x^i} - L^{\lambda}_{\alpha i}\dot{x}^{\alpha}\frac{\partial}{\partial \dot{x}^{\lambda}} \qquad (i = 1, \ldots\ldots, n), \tag{1}$$

where $\dot{x}^{\lambda}(\lambda = 1, \ldots, n)$ are the components of any vector.

Now let an r-parameter continuous transformation group be given by the following infinitesimal transformations

$$S_k = \xi^i_k(x)\frac{\partial}{\partial x^i} \qquad (k = 1, \ldots\ldots, r), \tag{2}$$

and any infinitesimal transformation belonging to this group be

$$S = \xi^i\frac{\partial}{\partial x^i} \tag{3}$$

If we denote the extended infinitesimal transformation of S, by

$$\dot{S} = \xi^i\frac{\partial}{\partial x^i} + \frac{\partial \xi^i}{\partial x^{\alpha}}\dot{x}^{\alpha}\frac{\partial}{\partial \dot{x}^i}, \tag{4}$$

then T_i $(i = 1, \ldots, n)$ are transformed by (3) as follows

$$T'_i = T_i + t(\dot{S}T_i) + \frac{t^2}{2!}\left(\dot{S}(\dot{S}T_i)\right) + \ldots\ldots \qquad (i = 1, \ldots\ldots, n).$$

The necessary and sufficient condition that the space whose parallelism of vectors is defined by (1), admits the group S in the extended sense, is that, for all values of t, T'_i $(i = 1, \ldots, n)$ must have the forms

$$\rho^l_i T_l + \varphi_i \dot{x}^{\lambda}\frac{\partial}{\partial \dot{x}^{\lambda}} \qquad (i = 1, \ldots\ldots, n), \tag{5}$$

where ρ^l_i $(i, l = 1, \ldots, n)$ are certain functions of x, and φ_i $(i = 1, \ldots, n)$ are arbitrary functions of x. Therefore, it must be that

$$(\dot{S}T_i) = \rho^l_i T_l + \varphi_i \dot{x}^{\lambda}\frac{\partial}{\partial \dot{x}^{\lambda}} \qquad (i = 1, \ldots\ldots n); \tag{6}$$

hence $\left(\dot{S}(\dot{S}T_i)\right)$, etc. can be expressed in the form of (5), and therefore T'_i $(i = 1, \ldots, n)$ also. But comparing the coefficients of $\frac{\partial}{\partial x^l}$ on both sides of (6), we have

$$\rho^l_i = -\frac{\partial \xi^l}{\partial x^i}.$$

Hence (6) becomes

$$(\dot{S}T_i) = -\frac{\partial \xi^l}{\partial x^i} T_l + \varphi_i \dot{x}^\lambda \frac{\partial}{\partial \dot{x}^\lambda} \qquad (i = 1, \ldots, n), \qquad (7)$$

where φ_i $(i = 1, \ldots, n)$ are arbitrary functions of x. So we have the result: The relations (7) are the necessary and sufficient conditions that the space admits the group S in the extended sense.

Further, comparing the coefficients of $\frac{\partial}{\partial \dot{x}^\lambda}$ on both sides of (7), we have

$$\frac{\partial^2 \xi^\alpha}{\partial x^i \partial x^j} + L^\alpha_{ik} \frac{\partial \xi^k}{\partial x^j} + L^\alpha_{kj} \frac{\partial \xi^k}{\partial x^i} + \xi^h \frac{\partial L^\alpha_{ij}}{\partial x^h} - L^k_{ij} \frac{\partial \xi^\alpha}{\partial x^k} = \delta^\alpha_i \varphi_j . \qquad (8)$$

Conversely, from (8) we can easily deduce (7). So we have the result: The relations (8) are the necessary and sufficient conditions that the space admits the group S in the extended sense.

If we denote the symmetric and the antisymmetric part of L^i_{jk} by Γ^i_{jk} and Ω^i_{jk} respectively, then the equations (8) are rewritten as follows:

$$\begin{cases} \dfrac{\partial^2 \xi^\alpha}{\partial x^i \partial x^j} + \Gamma^\alpha_{ik} \dfrac{\partial \xi^k}{\partial x^j} + \Gamma^\alpha_{kj} \dfrac{\delta \xi^k}{\partial x^i} + \xi^h \dfrac{\partial \Gamma^\alpha_{ij}}{\partial x^h} - \Gamma^k_{ij} \dfrac{\partial \xi^\alpha}{\partial x^k} = \delta^\alpha_i \varphi_j + \delta^\alpha_j \varphi_i & (9) \\ \Omega^\alpha_{ik} \dfrac{\partial \xi^k}{\partial x^j} + \Omega^\alpha_{kj} \dfrac{\partial \xi^k}{\partial x^i} + \xi^h \dfrac{\partial \Omega^\alpha_{ij}}{\partial x^h} - \Omega^k_{ij} \dfrac{\partial \xi^\alpha}{\partial x^k} = \delta^\alpha_i \varphi_j - \delta^\alpha_j \varphi_i . & (10) \end{cases}$$

Specially, when $\Omega^\alpha_{ij} = 0$, namely when the space is symmetric, we have from (10) $\varphi_i = 0$ $(i = 1, \ldots, n)$, and (9) becomes

$$\frac{\partial^2 \xi^\alpha}{\partial x^i \partial x_j} + \Gamma^\alpha_{ik} \frac{\partial \xi^k}{\partial x^j} + \Gamma^\alpha_{kj} \frac{\partial \xi^k}{\partial x^i} + \xi^h \frac{\partial \Gamma^\alpha_{ij}}{\partial x^h} - \Gamma^k_{ij} \frac{\partial \xi^\alpha}{\partial x^k} = 0 . \qquad (11)$$

This is not other than the condition that the space, the coefficients of connection being Γ^α_{ij}, admits the group S (not in the extended sense).(1) So we have

Theorem 1. When a symmetric space admits a group in the extended sense, then the space also admits the group (not in the extended sense).

(1) T. Sibata, *loc. cit.*, 113.

2. We shall next determine the conditions in which the number of the order of the group admitted by a space in the extended sense, becomes maximum. Using the method which Eisenhart adopted in his treatise[1], we see that this occurs when the Weyl tensor of Γ^{α}_{ij} of this space vanishes; and in this case by suitably choosing the coordinate system, φ_i and ξ^i $(i = 1, \ldots, n)$, which satisfy (9), become as follows

$$\varphi_i = a_i$$
$$\xi^i = a_h x^h x^i + b^i_l x^l + c^i \qquad (i = 1, \ldots, n) \tag{12}$$

where a's, b's, and c's are arbitrary constants.

Hence, if we choose a's, b's, and c's in (12) such that (12) satisfies (10) for a given Ω^{α}_{ij}, for all such a's, b's and c's (12) gives the greatest group which can be admitted by a space in the extended sense.

If we suppose that such a's, b's and c's are all arbitrary constants (12) would give the following n^2+2n infinitesimal transformations

$$\frac{\partial}{\partial x^i}, \quad x^l\frac{\partial}{\partial x^i}, \quad x^l x^j\frac{\partial}{\partial x^j} \qquad (i, l = 1, \ldots, n), \tag{13}$$

i.e. the general projective transformation group. By actual calculation we find that ξ^i corresponding to all the infinitesimal transformations of (13) does not satisfy (10) for each value of Ω^{α}_{ij}.

Hence the greatest group which can be admitted by a space in the extended sense, must be a sub-group of (13). From Lie's theorem[2], we know that the greatest sub-group of the general projective group (13) has $n(n+1)$ parameters, and they are similar either to the general linear group $\frac{\partial}{\partial x^i}$, $x^l\frac{\partial}{\partial x^i}$ $(i, l = 1, \ldots, n)$, or to $x^l\frac{\partial}{\partial x^i}$, $x^l x^j\frac{\partial}{\partial x^j}$ $(i, l = 1, \ldots, n)$. But by actual calculation we see that, for ξ^i corresponding to the later group, (8) doesnot hold for each value of L^i_{jk}, and for ξ^i corresponding to the former group, (8) holds when, and only when,

$$L^i_{jk} \overset{*}{=} \delta^i_j \psi_k$$

where ψ_k $(k = 1, \ldots, n)$ are the components of an arbitrary covariant vector. So we have

Theorem 2. The greatest group which can be admitted by a space in the extended sense, has $n(n+1)$ parameters and is similar to

(1) Eisenhart, *loc. cit.*, 126–131.

(2) S. Lie, *Theorie der Transformationsgruppe* **1** (1930), 569.

the general linear group $\frac{\partial}{\partial x^i}$, $x^l\frac{\partial}{\partial x^i}$ $(i, l = 1, \ldots, n)$, and in this case the coefficient of connection has the form

$$L^i_{jk} \overset{*}{=} \delta^i_j \psi_k ,$$

where ψ_k $(k = 1, \ldots, n)$ expresses an arbitrary covariant vector.

3. Lastly, in the general space which admits the group S in the extended sense, we will obtain the relations between the covariant derivative:

$$\nabla_i v^\lambda = \frac{\partial v^\lambda}{\partial x^i} + \Gamma^\lambda_{\alpha i} v^\alpha$$

and the transformation-derivative[1] by S:

$$\Delta v^\lambda = \xi^\alpha \frac{\partial v^\lambda}{\partial x^\alpha} - v^\alpha \frac{\partial \xi^\lambda}{\partial x^\alpha}$$

for an arbitrary vector field $v^\lambda(x)$. We have seen that the space defined by $T_1, \ldots, T_n$ (see (1)) admits the group S in the extended sense when, and only when,

$$(\dot{S}T_i) = -\frac{\partial \xi^l}{\partial x^i} T_l + \varphi_i \dot{x}^\lambda \frac{\partial}{\partial \dot{x}^\lambda} \quad (i = 1, \ldots, n). \qquad (7)$$

If we apply the operators on both sides of this relation, to an arbitrary system of equations of the form

$$f^\lambda \equiv \dot{x}^\lambda - v^\lambda(x) = 0 \qquad (\lambda = 1, \ldots, n),$$

and substituting $\dot{x}^\lambda = v^\lambda(x)$ $(\lambda = 1, \ldots, n)$ in the results, we have

$$\nabla_i \Delta v^\lambda - \Delta \nabla_i v^\lambda = \varphi_i v^\lambda . \qquad (14)$$

Conversely from (14) we can easily deduce the relations (7). So we have

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(1) T. Sibata, *loc. cit.*, 116, 117.

Theorem 3. A space admits a group in the extended sense when, and only when, for any arbitrary vector field $v^\lambda(x)$, the following relations hold

$$\nabla_i \nabla v^\lambda - \Delta \nabla_i v^\lambda = \varphi_i v^\lambda$$

where φ_i $(i = 1, \ldots, n)$ are arbitrary functions of x.