On Kernels and Spectra of Bounded Linear Transformations.

By

Fumitomo MAEDA.

(Received May 23, 1933.)

Let $\beta(E)$ be a completely additive, non-negative function of normal sets defined in a metric space R which is compact in itself, and be uniformly monotone almost everywhere (β)[1] in a β-normal set A. Let $\phi(E)$ be a complex valued set function which is absolutely continuous with respect to $\beta(E)$. When $\int_A |D_{\beta(E)}\phi(a)|^2 d\beta(E)$ is finite, then it is said that $\phi(E)$ belongs to the class $\mathfrak{L}_2(\beta)$. If we define the norm $\|\phi\|_\beta$ of $\phi(E)$ as

$$\|\phi\|_\beta = \left[\int_A |D_{\beta(E)}\phi(a)|^2 d\beta(E)\right]^{\frac{1}{2}},$$

and the inner product $(\phi, \psi)_\beta$ of two set functions $\phi(E)$ and $\psi(E)$ as

$$(\phi, \psi)_\beta = \int_A D_{\beta(E)}\phi(a)\overline{D_{\beta(E)}\psi(a)}d\beta(E),$$

then the space of all set functions of $\mathfrak{L}_2(\beta)$ is a Hilbert space.[2] I will denote this space also by $\mathfrak{L}_2(\beta)$.

In this paper, I first shew that all bounded linear transformations T defined in $\mathfrak{L}_2(\beta)$ are expressed in the integral form

$$T\phi(E) = \int_A D_{\beta(E')}\mathfrak{K}(E, a')D_{\beta(E')}\phi(a')d\beta(E'),$$

(1) That is, the β-value of the set of poles of $\beta(E)$ in A is zero. Cf. F. Maeda, this journal, **1** (1931), 3.

(2) Cf. my previous paper "On the Space of Real Set Functions," (this journal, **3** (1933), 1-42), where "linear manifolds" is equivalent to "closed linear manifolds" in this paper.

and find the explicit forms of the kernels $\mathfrak{K}(E, E')$.[1]

Next, to investigate the properties of the characteristic functions of the bounded linear transformations, I extend the normalized orthogonal system $\left\{\psi_\nu(E)\right\}$ which has positive integer ν as parameter, to the normalized orthogonal system $\left\{\Psi_{(U)}(E)\right\}$ which has set U as parameter. In respect of such normalized orthogonal systems I have several theorems to submit.

Then, I consider the spectra of the bounded linear transformations. Ordinarily, the resolution of identity $E(\lambda)$ is a function of point λ.[2] In this paper I define it as a function of set U, so that the characteristic functions of a bounded linear transformation depend upon the set U, which may be called the characteristic set of the transformation. These characteristic functions correspond to what is called, by Hellinger,[3] the "Eigendifferentialform" of a bounded quadratic form. Using the properties of set functions, I prove the theorems concerning characteristic functions, obtaining results analogous with those of Hellinger.[4]

In this paper, $\mathfrak{K}_{(E')}(E)$ means $\mathfrak{K}(E, E')$ considered as a function of set E, E' being a parameter. Similarly for $\mathfrak{K}_{(E)}(E')$. And when a series $\sum_\nu \phi_\nu(E)$ converges strongly to $\phi(E)$ in the space $\mathfrak{L}_2(\beta)$, I write as follows:

$$\phi(E)[=]_E \sum_\nu \phi_\nu(E) \qquad [\text{in } \mathfrak{L}_2(\beta)].$$

But, the words [in $\mathfrak{L}_2(\beta)$] are often omitted, when there is no possibility of confusion.

Kernel of Identical Transformation.

1. I first shew that $\beta(EE')$ is the kernel of identical transformation in $\mathfrak{L}_2(\beta)$.

---

(1) In the space of point functions of $\mathfrak{L}_2(\beta)$, some of the bounded linear transformations cannot be expressed in the integral form

$$Tf(a) = \int_A K(a, a')f(a')d\beta(E');$$

for example, the identical transformation. (Cf. J. v. Neumann, *Mathematische Grundlagen der Quantenmechanik*, (1932), 13.)

(2) Cf. J. v. Neumann, *loc. cit.*, 62.

(3) E. Hellinger, *Crelle's Journal*, **136** (1909), 242.

(4) *Ibid.*, 240–258.

Denote $\beta(EE')$ by $\mathfrak{E}(E, E')$. Then

$$D_{\beta(E')}\mathfrak{E}(E, a') = 1 \quad \text{when } a' \text{ is a point of } E,$$
$$= 0 \quad \text{when } a' \text{ is not a point of } E.^{(1)}$$

Hence
$$\int_A D_{\beta(E')}\mathfrak{E}(E, a')D_{\beta(E')}\phi(a')d\beta(E')$$
$$= \int_E D_{\beta(E')}\phi(a')d\beta(E') = \phi(E),$$

for all set functions $\phi(E)$ in $\mathfrak{L}_2(\beta)$. This equality shows that $\mathfrak{E}(E, E') = \beta(EE')$ *is the kernel of identical transformation.*

Kernels of Bounded Linear Transformations.

2. To find the kernels of bounded linear transformations, I will consider the general case, that is, transformations of set functions in $\mathfrak{L}_2(\beta)$ to set functions in $\mathfrak{L}_2(\sigma)$, where $\mathfrak{L}_2(\sigma)$ is defined as follows. Let S be a metric space which is compact in itself. S may be different from R, or coincide with R. And $\sigma(U)$ be a completely additive, non-negative function of normal sets defined in S, and be uniformly monotone almost everywhere (σ) in a σ-normal set V. Then a set function $\zeta(U)$ belongs to $\mathfrak{L}_2(\sigma)$, when $\zeta(U)$ is absolutely continuous with respect to $\sigma(U)$, and $\int_V |D_{\sigma(U)}\zeta(\lambda)|^2 d\sigma(U)$ is finite.

Let T be a bounded linear transformation, which transforms a set function $\phi(E)$ in $\mathfrak{L}_2(\beta)$ to a set function $\zeta(U)$ in $\mathfrak{L}_2(\sigma)$, and M_T be its modulus, so that

$$\zeta(U) = T\phi(U),$$

and
$$\|\zeta\|_\sigma \leqq M_T \|\phi\|_\beta .$$

Of course, if

$$[\lim_{n\to\infty}]\ \phi_n(E) = \phi(E),^{(2)}$$

then
$$[\lim_{n\to\infty}]\ T\phi_n(U) = T\phi(U). \tag{1}$$

(1) In this paper, I omit the words "almost everywhere (β)", and I use the symbol (=), as signifying "equal almost everywhere (β)".

(2) This means the strong convergence of $\{\phi_n(E)\}$ to $\phi(E)$.

Let $\{\psi_\nu(E)\}$ be a complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$. Put

$$\zeta_\nu(U) = T\psi_\nu(U) \qquad (\nu = 1, 2, \ldots) .$$

Now, I will shew that $\sum_\nu \zeta_\nu(U)\overline{\psi_\nu(E)}$ converges strongly as functions of set E and of set U.

Let $\{c_\nu\}$ be a sequence of complex numbers, such that $\sum_\nu |c_\nu|^2$ converges; and put

$$\phi(E)[=]_E \sum_\nu c_\nu \psi_\nu(E) .$$

Then, by (1),

$$T\phi(U)[=]_U \sum_\nu c_\nu \zeta_\nu(U) .$$

Therefore, for any set U, $\sum_\nu c_\nu \zeta_\nu(U)$ converges in the ordinary sense.(1) That is, $\sum_\nu c_\nu \zeta_\nu(U)$ always converges when $\sum_\nu |c_\nu|^2$ is convergent. Hence, by the converse of the Schwarzian inequality(2) $\sum_\nu |\zeta_\nu(U)|^2$ converges. Therefore $\sum_\nu \zeta_\nu(U)\overline{\psi_\nu(E)}$ converges strongly as functions of set E.

Next, expand $\mathfrak{E}_{(E')}(E) = \beta(EE')$ with respect to $\{\psi_\nu(E)\}$. Then, since

$$(\mathfrak{E}_{(E')}, \psi_\nu)_\beta = \int_A D_{\beta(E)} \mathfrak{E}(a, E') \overline{D_{\beta(E)} \psi_\nu(a)} d\beta(E)$$

$$= \int_{E'} \overline{D_{\beta(E)} \psi_\nu(a)} d\beta(E) = \overline{\psi_\nu(E')} ,$$

we have
$$\mathfrak{E}_{(E')}(E) [=]_E \sum_\nu \overline{\psi_\nu(E')}\, \psi_\nu(E) . \tag{2}$$

Apply T to $\mathfrak{E}_{(E')}(E)$, then we have

$$T\mathfrak{E}_{(E')}(U) [=]_U \sum_\nu \overline{\psi_\nu(E')}\, \zeta_\nu(U) .$$

Thus $\sum_\nu \zeta_\nu(U)\overline{\psi_\nu(E)}$ converges strongly as functions of set U and of set E. Therefore, it belongs to $\mathfrak{L}_2(\sigma)$ as a function of set U, and

(1) F. Maeda, this journal, **3** (1933), 4.
(2) P. Nalli, *Rendiconti di Palermo*, **46** (1922), 69.

belongs to $\mathfrak{L}_2(\beta)$ as a function of set E. Denote this function of two sets U and E by $\mathfrak{K}(U, E)$. Thus,

$$\mathfrak{K}(U, E)\,[=]_{U,E} \sum_{\nu} \zeta_\nu(U)\overline{\psi_\nu(E)}\,.^{(1)}$$

Let $\qquad \phi(E)\,[=]_E \sum_{\nu} c_\nu \psi_\nu(E)\,, \qquad c_\nu = (\phi, \psi_\nu)_\beta\,,$

be a set function in $\mathfrak{L}_2'(\beta)$, then

$$\int_A D_{\beta(E)}\mathfrak{K}(U, a) D_{\beta(E)}\phi(a)d\beta(E) = \sum_{\nu} \zeta_\nu(U)(\phi, \psi_\nu)_\beta$$
$$= \sum_{\nu} c_\nu \zeta_\nu(U) = T\phi(U)\,.$$

Hence $\mathfrak{K}(U, E)$ *is the required kernel of the bounded linear transformation* T.(2)

Thus all bounded linear transformations have their kernels. I will call these kernels the *bounded kernels*. Then, the bounded kernel $\mathfrak{K}(U, E)$ belongs to $\mathfrak{L}_2(\beta)$ as a function of set E, and its expansion with respect to a complete normalized orthogonal system $\{\overline{\psi_\nu(E)}\}$ is

$$\mathfrak{K}(U, E)\,[=]_E \sum_{\nu} \zeta_\nu(U)\overline{\psi_\nu(E)}\,,$$

where $\zeta_\nu(U)$ are set functions in $\mathfrak{L}_2(\sigma)$, and $\sum_{\nu} c_\nu \zeta_\nu(U)$ converges strongly when $\sum_{\nu} |c_\nu|^2$ converges. In this case, of course, as proved above,

$$\mathfrak{K}(U, E)\,[=]_U \sum_{\nu} \zeta_\nu(U)\overline{\psi_\nu(E)}\,,$$

and $\mathfrak{K}(U, E)$ belongs to $\mathfrak{L}_2(\sigma)$ as a function of set U.

When T is a bounded linear transformation which transforms a set function in the class $\mathfrak{L}_2(\beta)$ to a set function in the same clase $\mathfrak{L}_2(\beta)$, its kernel $\mathfrak{A}(E, E')$ may be obtained by putting $\phi_\nu(E)$ instead of $\zeta_\nu(U)$, where

(1) This means that $\sum_{\nu} \zeta_\nu(U)\overline{\psi_\nu(E)}$ converges strongly to $\mathfrak{K}(U, E)$ as functions of set U and of set E.

(2) $\mathfrak{K}(U, E)$ is essentially equivalent to what is called by M. Plancherel "fonction génératrice". (*Rendiconti di Palermo*, **30** (1910), 303). J. Radon obtained $\mathfrak{K}(U, E)$ from the standpoint of bilinear operations. (*Sitzgber. Akad. Wiss. Wien* II$_a$ **122** (1913), 1384).

$$\phi_\nu(E) = T\psi_\nu(E) \qquad (\nu = 1, 2, \ldots.) .$$

Thus $$\mathfrak{A}(E, E')\,[=]_{E, E'} \sum_\nu \phi_\nu(E)\overline{\psi_\nu(E')} .$$

In the special case when T is the identical transformation in $\mathfrak{L}_2(\beta)$, since

$$\phi_\nu(E) = \psi_\nu(E) \qquad (\nu = 1, 2, \ldots.) ,$$

its kernel must be $\sum_\nu \psi_\nu(E)\overline{\psi_\nu(E')}$, which is equal to $\beta(EE')$ by (2). Thus we have the same result as in sec. 1.

3. Put $$\mathfrak{K}^*(E, U)\,[=]_{E, U} \sum_\nu \psi_\nu(E)\overline{\zeta_\nu(U)} ,$$

and let $\zeta(U)$ be any set function in $\mathfrak{L}_2(\sigma)$. Then

$$\int_V D_{\sigma(U)}\mathfrak{K}^*(E, \lambda)D_{\sigma(U)}\zeta(\lambda)d\sigma(U) = \sum_\nu (\zeta, \zeta_\nu)_\sigma \psi_\nu(E) \qquad (1)$$

for any β-normal subset E of A. Next, I will shew that (1) converges strongly.

Let $$\chi_n(E) = (\zeta, \zeta_1)_\sigma\psi_1(E) + \ldots. + (\zeta, \zeta_n)_\sigma\psi_n(E) ,$$

then $$T\chi_n(U) = (\zeta, \zeta_1)_\sigma\zeta_1(U) + \ldots. + (\zeta, \zeta_n)_\sigma\zeta_n(U) .$$

Hence, $$(T\chi_n, \zeta)_\sigma = \sum_{\nu=1}^{n} |(\zeta, \zeta_\nu)_\sigma|^2 . \qquad (2)$$

On the other hand,

$$|(T\chi_n, \zeta)_\sigma| \leqq \|T\chi_n\|_\sigma \|\zeta\|_\sigma \leqq M_T \|\chi_n\|_\beta \|\zeta\|_\sigma$$

$$= M_T \sqrt{\sum_{\nu=1}^{n} |(\zeta, \zeta_\nu)_\sigma|^2}\ \|\zeta\|_\sigma . \qquad (3)$$

From (2) and (3)

$$\sum_{\nu=1}^{n} |(\zeta, \zeta_\nu)_\sigma|^2 \leqq M_T^2 \|\zeta\|_\sigma^2 , \qquad (4)$$

for any value of n. Then, since $\sum_{\nu=1}^{\infty} |(\zeta, \zeta_\nu)_\sigma|^2$ converges, (1) converges strongly.

Therefore, $\mathfrak{K}^*(E, U)$ is the kernel of a linear transformation which transforms any set function in $\mathfrak{L}_2(\sigma)$ to a set function in $\mathfrak{L}_2(\beta)$. Denote this transformation by T^*. Then, by (1) and (4),

$$\| T^*\zeta \|_\beta^2 = \sum_{\nu=1}^{\infty} | (\zeta, \zeta_\nu)_\sigma |^2 \leqq M_T^2 \| \zeta \|_\sigma^2 .$$

Hence, T^* is also a bounded linear transformation.

Let $\phi(E)$ and $\zeta(U)$ be any set functions in $\mathfrak{L}_2(\beta)$ and $\mathfrak{L}_2(\sigma)$ respectively. Then

$$T\phi(U) [=]_U \sum_\nu (\phi, \psi_\nu)_\beta \zeta_\nu(U) ,$$

$$T^*\zeta(E) [=]_E \sum_\nu (\zeta, \zeta_\nu)_\sigma \psi_\nu(E) .$$

Hence $$(T\phi, \zeta)_\sigma = \sum_\nu (\phi, \psi_\nu)_\beta (\zeta_\nu , \zeta)_\sigma = (\phi, T^*\zeta)_\beta .$$

Therefore, T^* is nothing else than the so-called *adjoint transformation* of T. Hence, the existence of an adjoint transformation for any bounded linear transformation is evident.[1]

4. Let $\mathfrak{K}(U, E)$ be a kernel of a bounded linear transformation which transforms a set function in $\mathfrak{L}_2(\beta)$ to a set function in $\mathfrak{L}_2(\sigma)$. Consider another class $\mathfrak{L}_2(\omega)$ of set functions, and let $\mathfrak{Q}(W, U)$ be a kernel of a bounded linear transformation which transforms a set function in $\mathfrak{L}_2(\sigma)$ to a set function in $\mathfrak{L}_2(\omega)$. Then, by sec. 2, $\mathfrak{K}(U, E)$ and $\mathfrak{Q}(W, U)$ are expressed as follows:

$$\mathfrak{K}(U, E) [=]_{U, E} \sum_\nu \zeta_\nu(U) \overline{\psi_\nu(E)} ,$$

$$\mathfrak{Q}(W, U) [=]_{W, U} \sum_\mu \rho_\mu(W) \overline{\eta_\mu(U)} ,$$

where $\{\psi_\nu(E)\}$ and $\{\eta_\mu(U)\}$ are complete normalized orthogonal systems in $\mathfrak{L}_2(\beta)$ and $\mathfrak{L}_2(\sigma)$ respectively, and

$$\zeta_\nu(U) = T_{\mathfrak{K}} \psi_\nu(U)^{(2)} \qquad (\nu = 1, 2, \ldots.) ,$$

$$\rho_\mu(W) = T_{\mathfrak{Q}} \eta_\mu(W) \qquad (\mu = 1, 2, \ldots.) .$$

(1) Cf. F. Riesz, *Acta Litterarum, Szeged*, **5** (1930), 29.

(2) $T_{\mathfrak{K}}$ means the linear transformation with kernel $\mathfrak{K}(U, E)$.

Put $$\int_V D_{\sigma(U)}\mathfrak{Q}(W, \lambda)D_{\sigma(U)}\mathfrak{K}(\lambda, E)d\sigma(U) = \mathfrak{R}(W, E), \qquad (1)$$

then, since $\overline{\mathfrak{R}(W, E)}$ may be considered as $T_{\mathfrak{K}}{}^*\overline{\mathfrak{Q}_{(W)}(E)}$, we have

$$\mathfrak{R}(W, E)\,[=]_E \sum_\nu (\mathfrak{Q}_{(W)}, \overline{\zeta_\nu})_\sigma\, \psi_\nu(E).$$

Put $$(\mathfrak{Q}_{(W)}, \overline{\zeta_\nu})_\sigma = \tau_\nu(W),$$

then, since

$$(\mathfrak{Q}_{(W)}, \overline{\zeta_\nu})_\sigma = \int_V D_{\sigma(U)}\mathfrak{Q}(W, \lambda)D_{\sigma(U)}\zeta_\nu(\lambda)d\sigma(U)$$
$$= T_{\mathfrak{Q}}\zeta_\nu(W),$$

we have $$\tau_\nu(W) = T_{\mathfrak{Q}}\zeta_\nu(W)\,[=]_W \sum_\mu (\zeta_\nu, \eta_\mu)_\sigma \rho_\mu(W), \qquad (2)$$

and $\tau_\nu(W)$ belongs to $\mathfrak{L}_2(\omega)$.

Let $\{c_\nu\}$ be a sequence of complex numbers, such that $\sum_\nu |c_\nu|^2$ converges, and put

$$\phi(E)\,[=]_E \sum_\nu c_\nu \psi_\nu(E).$$

Then $$T_{\mathfrak{K}}\phi(U)\,[=]_U \sum_\nu c_\nu \zeta_\nu(U),$$

and $$T_{\mathfrak{Q}}T_{\mathfrak{K}}\phi(W)\,[=]_W \sum_\nu c_\nu T_{\mathfrak{Q}}\zeta_\nu(W),$$

hence by (2) $$T_{\mathfrak{Q}}T_{\mathfrak{K}}\phi(W)\,[=]_W \sum_\nu c_\nu \tau_\nu(W). \qquad (3)$$

Therefore, by sec. 2,

$$\mathfrak{R}(W, E)\,[=]_E \sum_\nu \tau_\nu(W)\overline{\psi_\nu(E)}$$

is a bounded kernel. Then, since

$$\mathfrak{R}(W, E)\,[=]_W \sum_\nu \tau_\nu(W)\overline{\psi_\nu(E)},$$

we have $$\mathfrak{R}(W, E)\,[=]_{W,E} \sum_\nu \sum_\mu (\zeta_\nu, \eta_\mu)_\sigma \rho_\mu(W)\overline{\psi_\nu(E)}.^{(1)}$$

(1) If $\Phi^{(i)}(E, U)\,[=]_{E,U} \sum_\nu \Phi_\nu^{(i)}(E, U)$ and $\Phi(E, U)\,[=]_{E,U} \sum_i \Phi^{(i)}(E, U)$, then I will write $\Phi(E, U)\,[=]_{E,U} \sum_i \sum_\nu \Phi_\nu^{(i)}(E, U)$.

And (3) may be expressed as

$$T_{\mathfrak{Q}} T_{\mathfrak{K}} \phi(W) = T_{\mathfrak{N}} \phi(W) . \tag{4}$$

Now I will introduce the following abbreviation:

$$\int_A D_{\beta(E)} \mathfrak{K}(U, a) D_{\beta(E)} \phi(a) d\beta(E) = \mathfrak{K}\phi(U) ,$$

$$\int_V D_{\sigma(U)} \mathfrak{Q}(W, \lambda) D_{\sigma(U)} \mathfrak{K}(\lambda, E) d\sigma(U) = \mathfrak{Q}\mathfrak{K}(W, E) .$$

Then $$T_{\mathfrak{K}} \phi(U) = \mathfrak{K}\phi(U) ,$$

and (1) and (4) may be written as

$$\mathfrak{Q}\mathfrak{K}(W, E) = \mathfrak{N}(W, E) ,$$

$$\mathfrak{Q}\left\{\mathfrak{K}\phi(W)\right\} = \mathfrak{N}\phi(W) .$$

Hence, we have

$$\mathfrak{Q}\left\{\mathfrak{K}\phi(W)\right\} = \left\{\mathfrak{Q}\mathfrak{K}\right\}\phi(W) . \tag{5}$$

Let $\mathfrak{H}(E, A)$ be a kernel of a bounded linear transformation which transforms a set function in, say class $\mathfrak{L}_2(\gamma)$, to a set function in $\mathfrak{L}_2(\beta)$. And let $\mathfrak{H}_{(A)}(E)$ be put instead of $\phi(E)$ in (5). Then, since $\mathfrak{K}\mathfrak{H}_{(A)}(U)$ and $\left\{\mathfrak{Q}\mathfrak{K}\right\}\mathfrak{H}_{(A)}(W)$ are nothing but $\mathfrak{K}\mathfrak{H}(U, A)$ and $\left\{\mathfrak{Q}\mathfrak{K}\right\}\mathfrak{H}(W, A)$, we have

$$\mathfrak{Q}\left\{\mathfrak{K}\mathfrak{H}(W, A)\right\} = \left\{\mathfrak{Q}\mathfrak{K}\right\}\mathfrak{H}(W, A) . \tag{6}$$

Now, (5) and (6) correspond to

$$T_{\mathfrak{Q}} T_{\mathfrak{K}} \phi(W) = (T_{\mathfrak{Q}} T_{\mathfrak{K}}) \phi(W) ,$$

and $$T_{\mathfrak{Q}} T_{\mathfrak{K}} T_{\mathfrak{H}} = (T_{\mathfrak{Q}} T_{\mathfrak{K}}) T_{\mathfrak{H}} .$$

Hence, $\mathfrak{K}$ and $T_{\mathfrak{K}}$ obey the same rules of operation, and we can use $\mathfrak{K}$ instead of $T_{\mathfrak{K}}$. In what follows, let $\mathfrak{K}$ have two meanings, namely the bounded kernel $\mathfrak{K}(U, E)$ and the bounded linear transformation with kernel $\mathfrak{K}(U, E)$.

Extension of Normalized Orthogonal Systems.

5. Let $\{\psi_\nu(E)\}$ be a complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$. Then the condition of normalized orthogonality is

$$(\psi_\mu, \psi_\nu)_\beta = \delta_{\mu\nu}, \tag{1}$$

where $\qquad \delta_{\mu\nu} = 1 \qquad$ when $\quad \mu = \nu$,

$\qquad\qquad = 0 \qquad$ when $\quad \mu \neq \nu$.

And the condition of completeness in $\mathfrak{L}_2(\beta)$ is

$$\phi(E) \,[=]_E \sum_\nu c_\nu \psi_\nu(E), \tag{2}$$

where $\qquad c_\nu = (\phi, \psi_\nu)_\beta \qquad (\nu = 1, 2, \ldots\ldots)$,

for any set function $\phi(E)$ in $\mathfrak{L}_2(\beta)$.[1]

But, *the condition of completeness* (2) *can also be expressed as*

$$\sum_\nu \psi_\nu(E)\overline{\psi_\nu(E')} \,[=]_{E,E'}\, \beta(EE'). \tag{3}$$

For, put $\mathfrak{E}(E, E') = \beta(EE')$, and let

$$\phi(E) = \mathfrak{E}_{(E')}(E).$$

Then, since

$$(\mathfrak{E}_{(E')}, \psi_\nu)_\beta = \int_A D_{\beta(E)}\mathfrak{E}_{(E')}(a)\overline{D_{\beta(E)}\psi_\nu(a)}d\beta(E)$$
$$= \int_{E'} \overline{D_{\beta(E)}\psi_\nu(a)}d\beta(E)^{(2)} = \overline{\psi_\nu(E')},$$

we have, by (2)

$$\beta(EE') \,[=]_E \sum_\nu \psi_\nu(E)\overline{\psi_\nu(E')}.$$

---

(1) Cf. J. v. Neumann, *loc. cit.*, 28.

(2) Cf. sec. 1.

From the symmetric property of E and E', we have

$$\beta(EE')\,[=]_{E'}\,\sum_{\nu}\psi_{\nu}(E)\overline{\psi_{\nu}(E')}\,.$$

That is, (3) follows from (2).

Next, from (3) we have

$$\phi(E)=\mathfrak{E}\phi(E)=\sum_{\nu}(\phi,\psi_{\nu})_{\beta}\psi_{\nu}(E)\,. \tag{4}$$

But, $\left\{\psi_{\nu}(E)\right\}$ being normalized orthogonal system, it is obvious that (4) converges strongly. Hence, we have (2) from (3).

$\left\{\psi_{\nu}(E)\right\}$ is a system, whose parameter is a positive integer ν. Now, I proceed to extend this system to one whose parameter is a set U.

Let $\Psi(E, U)$ be a bounded kernel, then if we consider U as parameter, $\left\{\Psi_{(U)}(E)\right\}$ is a system of set functions in $\mathfrak{L}_2(\beta)$. Corresponding to (1) and (3), we may give the conditions of normalized orthogonality and completeness of $\left\{\Psi_{(U)}(E)\right\}$ as follows:

Condition of normalized orthogonality:

$$(\Psi_{(U)}\,,\,\Psi_{(U')})_{\beta}=\sigma(UU')\,, \tag{5}$$

$\sigma(U)$ may be called the *base* of the system $\left\{\Psi_{(U)}(E)\right\}$.

Condition of completeness in $\mathfrak{L}_2(\beta)$*:*

$$(\Psi_{(E)}\,,\,\Psi_{(E')})_{\sigma}=\beta(EE').^{(1)} \tag{6}$$

By reason of the symmetrical property of (5) and (6) with respect to E and U, the complete normalized orthogonality of the system $\left\{\Psi_{(U)}(E)\right\}$ in $\mathfrak{L}_2(\beta)$ with base $\sigma(U)$, is nothing but the complete normalized orthogonality of the system $\left\{\Psi_{(E)}(U)\right\}$ in $\mathfrak{L}_2(\sigma)$ with base $\beta(E)$, the two conditions being interchanged.

Corresponding to (2), we may put

$$\phi(E)=\int_V D_{\sigma(U)}\Psi(E,\lambda)D_{\sigma(U)}\zeta(\lambda)d\sigma(U)^{(2)} \tag{7}$$

where $$\zeta(U)=\int_A\overline{D_{\beta(E)}\Psi(a,U)}D_{\beta(E)}\phi(a)d\beta(E)\,.$$

(1) The integration with respect to $\sigma(U)$ corresponds to the summation with respect to ν.

(2) This is the extension of the so-called integral representation of functions. Cf. Plancherel, *loc. cit.*, 297.

If we use the abbreviated form, (7) may be expressed as follows:

$$\phi(E) = \Psi\zeta(E)$$

where
$$\zeta(U) = \Psi^*\phi(U)\ .$$

But, this is obvious, for, since from (6)

$$\Psi\Psi^*(E, E') = (\Psi_{(E)}, \Psi_{(E')})_\sigma = \beta(EE')\ ,$$

therefore we have

$$\phi(E) = (\Psi\Psi^*)\phi(E) = \Psi\Psi^*\phi(E)\ .$$

6. *If $\{\Psi_{(U)}(E)\}$ is a normalized orthogonal system in $\mathfrak{L}_2(\beta)$ with base $\sigma(U)$, then $\Psi(E, U)$ is expressed as follows:*

$$\Psi(E, U)\ [=]_{E,U} \sum_\nu \psi_\nu(E)\overline{\eta_\nu(U)}\ ,$$

where $\{\eta_\nu(U)\}$ is a complete normalized orthogonal system in $\mathfrak{L}_2(\sigma)$, and $\{\psi_\nu(E)\}$ is a normalized orthogonal system in $\mathfrak{L}_2(\beta)$.

If, in addition, $\{\Psi_{(U)}(E)\}$ is complete in $\mathfrak{L}_2(\beta)$, then $\{\psi_\nu(E)\}$ is complete in $\mathfrak{L}_2(\beta)$.

For, by sec. 2, we can expand the bounded kernel $\Psi(E, U)$ with respect to the complete normalized orthogonal system $\{\overline{\eta_\nu(U)}\}$ in $\mathfrak{L}_2(\sigma)$, i.e.

$$\Psi(E, U)\ [=]_{E,U} \sum_\nu \psi_\nu(E)\overline{\eta_\nu(U)}\ , \tag{1}$$

where
$$\psi_\nu(E) = \Psi\eta_\nu(E) \qquad (\nu = 1, 2, \dots\dots)\ .$$

Then, by sec. 3,

$$(\psi_\mu, \psi_\nu)_\beta = (\Psi\eta_\mu, \Psi\eta_\nu)_\beta = (\eta_\mu, \Psi^*\Psi\eta_\nu)_\sigma\ .$$

But, by the normalized orthogonality of $\{\Psi_{(U)}(E)\}$

$$\Psi^*\Psi(U, U') = (\Psi_{(U')}, \Psi_{(U)})_\beta = \sigma(UU')\ ,$$

we have
$$\Psi^*\Psi\eta_\nu(U) = \eta_\nu(U)\ .$$

Hence
$$(\psi_\mu, \psi_\nu)_\beta = (\eta_\mu, \eta_\nu)_\sigma = \delta_{\mu\nu}\ .$$

That is, $\left\{\psi_\nu(E)\right\}$ is a normalized orthogonal system in $\mathfrak{L}_2(\beta)$.

Now, by (1), we have

$$(\Psi_{(E)}, \Psi_{(E')})_\sigma [=]_{E, E'} \sum_\nu \psi_\nu(E)\overline{\psi_\nu(E')} .$$

If $\left\{\Psi_{(U)}(E)\right\}$ is complete in $\mathfrak{L}_2(\beta)$, then

$$(\Psi_{(E)}, \Psi_{(E')})_\sigma = \beta(EE') .$$

Hence, by (3) of the preceding section, $\left\{\psi_\nu(E)\right\}$ is complete in $\mathfrak{L}_2(\beta)$.

7. *Let $\left\{\Phi_{(U)}(E)\right\}$ be a system of set functions in $\mathfrak{L}_2(\beta)$, and let $\Phi_{(E)}(U)$ be a completely additive function of σ-normal set U. If*

$$(\Phi_{(U)}, \Phi_{(U')})_\beta = \sigma(UU') , \tag{1}$$

then $\left\{\Phi_{(U)}(E)\right\}$ is a normalized orthogonal system with base $\sigma(U)$.

To prove this theorem, it is sufficient to shew that $\Phi(E, U)$ is a bounded kernel.

Let $\phi(E)$ be any set functicn ie $\mathfrak{L}_2(\beta)$. Then $(\phi, \Phi_{(U)})_\beta$ is a function of U. Denote this function by $\zeta(U)$.

Let $$U = U_1 + U_2 + \ldots\ldots + U_i + \ldots\ldots ,$$

then, by (1)

$$(\Phi_{(U_i)}, \Phi_{(U_j)})_\beta = 0 \qquad \text{when} \quad i \neq j .$$

Hence, $$\Phi_{(U)}(E) [=]_E \sum_i \Phi_{(U_i)}(E) .$$

Therefore, $$(\phi , \Phi_{(U)})_\beta = \sum_i (\phi, \Phi_{(U_j)})_\beta .$$

That is, $\zeta(U)$ is completely additive.

By (1), $$\| \Phi_{(U)} \|_\beta^2 = \sigma(U) ,$$

hence, for a set U, where $\sigma(U) = 0$, $\Phi_{(U)}(E)$ is a null function; therefore $\zeta(U) = 0$. That is, $\zeta(U)$ is absolutely continuous with respect to $\sigma(U)$.

Divide V into the sum of σ-normal sets, i.e.

$$V = U_1 + U_2 + \ldots\ldots + U_i + \ldots\ldots .$$

Then, from (1) $\left\{\frac{\Phi_{(U_i)}(E)}{\sqrt{\sigma(U_i)}}\right\}$ $(i=1, 2, \ldots\ldots)$ is a normalized orthogonal system in $\mathfrak{L}_2(\beta)$. Moreover the coefficients of the expansion of $\phi(E)$ with respect to this system are

$$\left(\phi, \frac{\Phi_{(U_i)}}{\sqrt{\sigma(U_i)}}\right)_\beta = \frac{\zeta(U_i)}{\sqrt{\sigma(U_i)}} \qquad (i=1, 2, \ldots\ldots).$$

Hence, by Bessel's inequality[1]

$$\sum_i \frac{|\zeta(U_i)|^2}{\sigma(U_i)} \leqq \|\phi\|_\beta^2 .$$

But, since this inequality holds for any division of V, $\zeta(U)$ belongs to $\mathfrak{L}_2(\sigma)$, and

$$\|\zeta\|_\sigma \leqq \|\phi\|_\beta .\ ^{(2)}$$

Hence

$$\zeta(U) = (\phi, \Phi_{(U)})_\beta = \int_A D_{\beta(E)}\, \Phi^*(U, a) D_{\beta(E)}\phi(a) d\beta(E)$$

is a bounded linear transformation which transforms a set function $\phi(E)$ in $\mathfrak{L}_2(\beta)$ to a set function $\zeta(U)$ in $\mathfrak{L}_2(\sigma)$. That is, $\Phi(E, U)$ is a bounded kernel.

8. *Lel $\phi(E)$ be a set function in $\mathfrak{L}_2(\beta)$, and $\{\psi_\nu(E)\}$ be a complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$. Put*

$$\phi_\nu(E) = \int_E D_{\beta(E)}\psi_\nu(a) \overline{D_{\beta(E)}\phi(a)} d\beta(E)\ , \tag{1}$$

then $\{\phi_\nu(E)\}$ is a complete normalized orthogonal system in $\mathfrak{L}_2(\gamma)$, where

$$\gamma(E) = \int_E |D_{\beta(E)}\phi(a)|^2 d\beta(E)\ .$$

To prove this theorem, put

$$\mathfrak{B}(E, E') = \phi(EE')\ .$$

(1) F. Maeda, this journal, **3** (1933), 10.

(2) Cf. E. W. Hobson, *The theory of functions of a real variable*, I, third ed. (1927), 670.

Then
$$D_{\beta(E)}\mathfrak{B}(a, E') = D_{\beta(E)}\phi(a) \quad \text{when } a \text{ is a point of } E',$$
$$= 0 \qquad \text{when } a \text{ is not a point of } E'.$$

Hence, $\left\{\mathfrak{B}_{(E')}(E)\right\}$ is a system of set functions in $\mathfrak{L}_2(\beta)$, and

$$\int_A D_{\beta(E'')}\mathfrak{B}(a'', E)\overline{D_{\beta(E'')}\mathfrak{B}(a'', E')}d\beta(E'')$$
$$= \int_{EE'} |D_{\beta(E)}\phi(a)|^2 d\beta(E) = \gamma(EE'). \qquad (2)$$

Then, by sec. 7, $\left\{\mathfrak{B}_{(E')}(E)\right\}$ is a normalized orthogonal system in $\mathfrak{L}_2(\beta)$ with base $\gamma(E')$. (1)

Now, by (2) $||\mathfrak{B}_{(E')}||_\beta^2 = \gamma(E')$, and hence $\mathfrak{B}_{(E)}(E')$ is absolutely continuous with respect to $\gamma(E')$. And

$$D^{(s)}_{\gamma(E')}\mathfrak{B}(E, a') = \frac{D^{(s)}_{\beta(E')}\mathfrak{B}(E, a')}{D^{(s)}_{\beta(E')}\gamma(a')}$$

$$\begin{cases} = \dfrac{D_{\beta(E')}\phi(a')}{|D_{\beta(E')}\phi(a')|^2} = \dfrac{1}{D_{\beta(E')}\phi(a')} & \text{when } a' \text{ is a point of } E, \\ = 0 & \text{when } a' \text{ is not a point of } E. \end{cases}$$

---

(1) A is γ-normal, and $\gamma(E)$ is uniformly monotone almost everywhere (γ) in A. For, since $\gamma(E)$ is absolutely continuous with respect to $\beta(E)$, and A is β-normal, it is evident that A is γ-normal. Next, a finite symmetric derivative $D^{(s)}_{\beta(E)}\gamma(a)$ exists at all points of A, except at these points of H, where $\beta(H) = 0$. (Cf. F. Maeda, this journal, **1** (1931), 11.) Let a be a point in $A-H$. Then

$$\lim_{\rho\to 0}\left[\frac{\gamma\{\bar{U}(a,\rho)\}}{\beta\{\bar{U}(a,\rho)\}}\Big/\frac{\gamma\{\bar{U}(a,\lambda\rho)\}}{\beta\{\bar{U}(a,\lambda\rho)\}}\right] = 1,$$

where $\bar{U}(a, \rho)$ is a closed neighbourhood of a with radius ρ, and $\lambda > 1$. Therefore

$$\lim_{\rho\to 0}\frac{\gamma\{\bar{U}(a,\rho)\}}{\gamma\{\bar{U}(a,\lambda\rho)\}} = \lim_{\rho\to 0}\frac{\beta\{\bar{U}(a,\rho)\}}{\beta\{\bar{U}(a,\lambda\rho)\}},$$

Hence, $\beta(E)$ being uniformly monotone almost everywhere (β) in A, $\gamma(E)$ is uniformly monotone almost everywhere (γ) in A. (Cf. *ibid.*, 3.) Since any β-normal set is γ-normal, $\mathfrak{B}_{(E)}(E')$ is a completely additive function of γ-normal sets.

Hence $$\int_A D_{\gamma(E'')}\mathfrak{B}(E, a'')\overline{D_{\gamma(E'')}\mathfrak{B}(E', a'')}d\gamma(E'')$$

$$= \int_{EE'} \frac{1}{|D_{\beta(E'')}\phi(a'')|^2} d\gamma(E'') = \int_{EE'} d\beta(E'')^{(1)} = \beta(EE').$$

Therefore, by sec. 5, $\left\{\mathfrak{B}_{(E')}(E)\right\}$ is complete in $\mathfrak{L}_2(\beta)$.

Then, by sec. 6,

$$\mathfrak{B}(E, E')\,[=]_{E,E'} \sum_\nu \psi_\nu(E)\overline{\phi'_\nu(E')} \quad [\text{in } \mathfrak{L}_2(\beta) \text{ and } \mathfrak{L}_2(\gamma)], \tag{3}$$

where $\left\{\psi_\nu(E)\right\}$ and $\left\{\phi'_\nu(E')\right\}$ are complete normalized orthogonal systems in $\mathfrak{L}_2(\beta)$ and $\mathfrak{L}_2(\gamma)$ respectively.

But, since by (3)

$$\phi'_\nu(E') = \mathfrak{B}^* \psi_\nu(E') = \int_A \overline{D_{\beta(E)}\mathfrak{B}(a, E')} D_{\beta(E)}\psi_\nu(a)d\beta(E)$$

$$= \int_{E'} \overline{D_{\beta(E)}\phi(a)} D_{\beta(E)}\psi_\nu(a)d\beta(E) = \phi_\nu(E'),$$

therefore, $\left\{\phi_\nu(E)\right\}$ is a complete normalized orthogonal system in $\mathfrak{L}_2(\gamma)$.

---

(1) By the following theorem: *Let $\gamma(E)$ be a set function which is absolutely continuous with respect to $\beta(E)$, then*

$$\int_E f(a)d\gamma(E) = \int_E f(a)D_{\beta(E)}\gamma(a)d\beta(E), \tag{1}$$

when one of the two integrals exists.

By the footnote of the preceding page, $\gamma(E)$ is uniformly monotone almost everywhere (γ) in A. Assume that $\int_E f(a)d\gamma(E)$ exists, and is equal to $\chi(E)$. Then $\chi(E)$ is absolutely continuous with respect to $\beta(E)$. Since $D^{(s)}_{\gamma(E)}\chi(a) = \dfrac{D^{(s)}_{\beta(E)}\chi(a)}{D^{(s)}_{\beta(E)}\gamma(a)}$ almost everywhere (β), we have

$$\chi(E) = \int_E D^{(s)}_{\beta(E)}\chi(a)\,d\beta(E) = \int_E D^{(s)}_{\gamma(E)}\chi(a)D^{(s)}_{\beta(E)}\gamma(a)d\beta(E).$$

Now $D^{(s)}_{\gamma(E)}\chi(a) = f(a)$ except for the points of set H, where $\gamma(H) = 0$. But $D^{(s)}_{\beta(E)}\gamma(a) = 0$ almost everywhere (β) in H; hence

$$\chi(E) = \int_E f(a)D_{\beta(E)}\gamma(a)d\beta(E).$$

By a similar method, we can prove (1), when $\int_E f(a)D_{\beta(E)}\gamma(a)d\beta(E)$ exists.

Spectra of Bounded Linear Transformations.

9. When a bounded kernel $\mathfrak{A}(E, E')$ is equal to $\mathfrak{A}^*(E, E')$, then I will call $\mathfrak{A}(E, E')$ a *self-adjoint kernel*. In this case, since

$$(\mathfrak{A}\phi, \phi)_\beta = (\phi, \mathfrak{A}\phi)_\beta ,$$

$(\mathfrak{A}\phi, \phi)_\beta$ is real. Let the upper and lower bounds of $(\mathfrak{A}\psi, \psi)_\beta$ for all normalized set functions $\psi(E)$ in $\mathfrak{L}_2(\beta)$ be M and m respectively. Then, as F. Riesz proved,(1) corresponding to any bounded Baire's function $f(\lambda)$ defined in the interval $m \leqq \lambda \leqq M$, a bounded self-adjoint transformation $f(\mathfrak{A})$, and therefore a bounded self-adjoint kernel $f(\mathfrak{A})(E, E')$, exists.

Denote the closed interval $[m, M]$ by I. Let $h_{(U)}(\lambda) = h(\lambda, U)$ be de la Vallée Poussin's characteristic function of Borel subset U of I. That is,

$$h(\lambda, U) = 1 \qquad \text{when } \lambda \text{ is a point of } U,$$
$$= 0 \qquad \text{when } \lambda \text{ is not a point of } U.$$

Then, obviously $h_{(\lambda)}(U)$ is a completely additive set function defined for all Borel subsets of I. On the other hand, $h_{(U)}(\lambda)$ is a Baire's function defined in I. Hence a bounded self-adjoint transformation $h_{(U)}(\mathfrak{A})$ exists.

From the definition,

$$h_{(U)}(\lambda)h_{(U')}(\lambda) = h_{(UU')}(\lambda) ;$$

hence
$$h_{(U)}(\mathfrak{A})h_{(U')}(\mathfrak{A}) = h_{(UU')}(\mathfrak{A}) .$$

Then, since
$$[h_{(U)}(\mathfrak{A})]^2 = h_{(U)}(\mathfrak{A}) ,$$

$h_{(U)}(\mathfrak{A})$ is a projecting transformation, which transforms all set functions in $\mathfrak{L}_2(\beta)$ to their components contained in a closed linear manifold.(2) Since this closed linear manifold depends to U, I will denote it by $\mathfrak{M}_{(U)}$. That is, let $\{\psi_\nu(E)\}$ be a complete normalized orthogonal system in $\mathfrak{M}_{(U)}$, and let $\phi(E)$ be any set function in $\mathfrak{L}_2(\beta)$, then

$$h_{(U)}(\mathfrak{A})\phi(E) \, [=]_E \sum_\nu (\phi, \psi_\nu)_\beta \psi_\nu(E) .$$

(1) F. Riesz, *loc. cit.*, 31–36.
(2) Cf. J. v. Neumann, *loc. cit.*, 41.

If we denote this set function $h_{(U)}(\mathfrak{A})\phi(E)$ by $\Phi(E, U)$, then $\Phi_{(U)}(E)$ is a set function in $\mathfrak{M}_{(U)}$, and $\Phi_{(E)}(U)$ is a completely additive set function defined for all Borel subsets of I.

Let U_0 be a Borel subset of I, such that $h_{(U_0)}(\mathfrak{A})$ does not vanish identically. If $h_{(O)}(\mathfrak{A})$ does not vanish identically for all open subsets O of U_0, then I will call U_0 the *characteristic set* of $\mathfrak{A}$, and the set functions of $\mathfrak{M}_{(U_0)}$ the *characteristic functions* of $\mathfrak{A}$ with respect to the characteristic set U_0.

If the characteristic set U_0 is composed with a single point λ_0, then λ_0 belongs to the so-called *point spectrum* or *discontinuous spectrum.* The other points of the characteristic sets constitute the so-called *line spectrum* or *continuous spectrum.*

Let $\Phi(E, U)$ and $\Psi(E, U')$ be any set functions belonging to $\mathfrak{M}_{(U)}$ and $\mathfrak{M}_{(U')}$ respectively. Then

$$\begin{aligned}(\Phi_{(U)}, \Psi_{(U')})_\beta &= (h_{(U)}(\mathfrak{A})\Phi_{(U)}, h_{(U')}(\mathfrak{A})\Psi_{(U')})_\beta \\ &= (\Phi_{(U)}, h_{(U)}(\mathfrak{A})\, h_{(U')}(\mathfrak{A})\Psi_{(U')})_\beta \\ &= (\Phi_{(U)}, h_{(UU')}(\mathfrak{A})\Psi_{(U')})_\beta .\end{aligned}$$

Hence, if U and U' have no point in common, then $\Phi_{(U)}(E)$ and $\Psi_{(U')}(E)$ are orthogonal. That is, $\mathfrak{M}_{(U)}$ and $\mathfrak{M}_{(U')}$ are orthogonal.

Hence, the closed linear manifolds which correspond to the point spectrum, are orthogonal to each other. The dimension of the space $\mathfrak{L}_2(\beta)$ being denumerably infinite, *the point spectrum is, at most, denumerably infinite.*

10. Next, I proceed to find the relations between the spectra and the characteristic functions.

From the definition of the integral, it is evident that, for any bounded Baire's function $f(\lambda)$,

$$\int_I f(\lambda)\, d_U h(\mu, U) = f(\mu) ,$$

and

$$\int_{U_0} \lambda\, d_U h(\mu, U)^{(1)} = \mu h(\mu, U_0) ,$$

(1) Here λ is a point function, whose value is λ at point λ.

where $\int \ldots . d_U h(\mu, U)$ signifies the integration by $h(\mu, U)$ as a function of set U. Corresponding to these expressions, we have

$$\int_I f(\lambda)\, d_U h(\mathfrak{A}, U) = f(\mathfrak{A}), \tag{1}$$

and
$$\int_{U_0} \lambda\, d_U h(\mathfrak{A}, U) = \mathfrak{A}\, h(\mathfrak{A}, U_0). \tag{2}$$

Let $f(\lambda)$ be λ and 1; then we have from (1)

$$\int_I \lambda\, d_U h(\mathfrak{A}, U) = \mathfrak{A}, \tag{3}$$

$$\int_I d_U h(\mathfrak{A}, U) = \mathfrak{E}.$$

Hence, $h(\mathfrak{A}, U)$ corresponds to the so-called resolution of identity.(1)

Let $\phi(E)$ be any set function in $\mathfrak{L}_2(\beta)$, and put

$$h_{(U)}(\mathfrak{A})\phi(E) = \Phi(E, U).$$

If we apply (2) to $\phi(E)$, then we have

$$\int_{U_0} \lambda\, d_U \Phi(E, U) = \mathfrak{A}\, \Phi(E, U_0). \tag{4}$$

When, especially, U_0 is a point λ_0, (4) becomes:

$$\lambda_0\, \Phi(E, \lambda_0) = \mathfrak{A}\Phi(E, \lambda_0).$$

Hence λ_0 is the so-called characteristic constant of $\mathfrak{A}$, and $\Phi(E, \lambda_0)$ is the characteristic function of $\mathfrak{A}$ with respect to λ_0.

When U_0 is not a point, $\Phi(E, U_0)$ corresponds to what is called by Hellinger(2) the "Eigendifferentialform".

When U_1 and U_2 have no point in common, then by sec. 9,

$$(\Phi_{(U_1)}, \Phi_{(U_2)})_\beta = 0.$$

Hence
$$\begin{aligned}(\Phi_{(U)}, \Phi_{(U')})_\beta &= (\Phi_{(UU')} + \Phi_{(U-UU')}, \Phi_{(UU')} + \Phi_{(U'-UU')})_\beta \\ &= (\Phi_{(UU')}, \Phi_{(UU')})_\beta.\end{aligned}$$

(1) Cf. J. v. Neumann, *loc. cit.*, 62.

(2) E. Hellinger, *loc. cit.*, 242.

That is, $(\Phi_{(U)}, \Phi_{(U')})_\beta$ is a function of set UU'; denote this function by $\sigma(UU')$, namely,

$$(\Phi_{(U)}, \Phi_{(U')})_\beta = \sigma(UU') . \tag{5}$$

When $$U = U_1 + U_2 + \dots + U_i + \dots ,$$

then, since

$$(\Phi_{(U_i)}, \Phi_{(U_j)})_\beta = \sigma(U_i) \qquad \text{when} \quad i = j ,$$
$$= 0 \qquad \text{when} \quad i \neq j ,$$

we have $$\sigma(U) = (\Phi_{(U)}, \Phi_{(U)})_\beta = \sum_i (\Phi_{(U_i)}, \Phi_{(U_i)})_\beta = \sum_i \sigma(U_i) .$$

That is, $\sigma(U)$ is a completely additive, non-negative set function defined for all Borel subsets of I.

Since $\Phi(E, U)$ satisfies (5), by sec. 7, $\{\Phi_{(U)}(E)\}$ is a normalized orthogonal system with base $\sigma(U)$.[1] Hence, by sec. 6, *$\Phi(E, U)$ is expressed as follows:*

$$\Phi(E, U) [=]_{E, U} \sum_\nu \psi_\nu(E) \overline{\eta_\nu(U)} ,$$

where $\{\eta_\nu(U)\}$ is a complete normalized orthogonal system in $\mathfrak{L}_2(\sigma)$, and $\{\psi_\nu(E)\}$ is a normalized orthogonal system in $\mathfrak{L}_2(\beta)$.

Denote the closed linear manifold, whose fundamental system is $\{\psi_\nu(E)\}$, by $\mathfrak{M}_\Phi$.

11. Let $\Phi_1(E, U)$ be a characteristic function of $\mathfrak{A}$ with base $\sigma_1(U)$, then, as seen in the preceding section, we have the following expression:

$$\Phi_1(E, U) [=]_{E, U} \sum_\nu \psi_\nu^{(1)}(E) \overline{\eta_\nu^{(1)}(U)} . \tag{1}$$

(1) Since $\sigma(U)$ is a set function defined in the Euclidian space, the uniform monotonity of $\sigma(U)$ at I has no bearing on the problem. For, in the Euclidian space, we can prove the fundamental theorem between integration and differentiation without the condition of uniform monotonity. Cf. F. Maeda, this journal, **2** (1932), 33.

Let $\left\{\psi'_\nu(E)\right\}$ be a complementary normalized orthogonal system in $\mathfrak{L}_2(\beta)$ with respect to $\left\{\psi^{(1)}_\nu(E)\right\}$.(1) Then, by sec. 2 we have

$$\mathfrak{A}(E, E')\,[=]_{E,E'} \sum_\nu \phi^{(1)}_\nu(E)\overline{\psi^{(1)}_\nu(E')} + \sum_\nu \phi'_\nu(E)\overline{\psi'_\nu(E')}\,,$$

where
$$\phi^{(1)}_\nu(E) = \mathfrak{A}\,\psi^{(1)}_\nu(E)\,, \qquad\qquad (\nu = 1, 2, \ldots\ldots).$$
$$\phi'_\nu(E) = \mathfrak{A}\,\psi'_\nu(E)\,.$$

From (1), we have

$$\Phi_1\,\Phi_1^*(E, E')\,[=]_{E,E'} \sum_\nu \psi^{(1)}_\nu(E)\overline{\psi^{(1)}_\nu(E')}\,,$$

hence
$$\mathfrak{A}\,\Phi_1\,\Phi_1^*(E, E')\,[=]_{E,E'} \sum_\nu \phi^{(1)}_\nu(E)\overline{\psi^{(1)}_\nu(E')}\,.$$

Therefore $\mathfrak{A}\,\Phi_1\,\Phi_1^*(E, E')$ is a bounded kernel.

But, since

$$\mathfrak{A}\Phi_1(E, U) = \int_U \lambda\, d_U \Phi_1(E, U)$$
$$= \int_U \lambda D_{\sigma_1(U)}\Phi_1(E, \lambda)d\sigma_1(U),^{(2)}$$

we have

$$\mathfrak{A}\,\Phi_1\,\Phi_1^*(E, E') = \int_I \lambda D_{\sigma_1(U)}\,\Phi_1(E, \lambda)\overline{D_{\sigma_1(U)}\Phi_1(E', \lambda)}d\sigma_1(U)\,. \qquad (2)$$

Therefore $\mathfrak{A}\,\Phi_1\,\Phi_1^*(E, E')$ is a self-adjoint kernel.

Then $\quad \mathfrak{A}(E, E') - \mathfrak{A}\,\Phi_1\,\Phi_1^*(E, E')\,[=]_{E,E'} \sum_\nu \phi'_\nu(E)\overline{\psi'_\nu(E')}$

is a bounded self-adjoint kernel. Denote this kernel by $\mathfrak{A}_1(E, E')$. Then, since $\mathfrak{A}_1\,\Phi_1(E, U) = 0$, $\Phi_1(E, U)$ is not a characteristic function of $\mathfrak{A}_1$. Let $\Phi_2(E, U)$ be a characteristic function of $\mathfrak{A}_1$ with base $\sigma_2(U)$. That is,

$$\int_U \lambda\, d_U\,\Phi_2(E, U) = \mathfrak{A}_1\,\Phi_2(E, U)\,. \qquad (3)$$

(1) This means that

$\psi^{(1)}_1(E),\ \psi^{(1)}_2(E),\ \ldots\ldots,\ \psi^{(1)}_\nu(E),\ \ldots\ldots,\ \psi'_1(E),\ \psi'_2(E),\ \ldots\ldots,\ \psi'_\nu(E),\ \ldots\ldots$

is a complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$.

(2) Cf. footnote, p. 258.

Then, since $\mathfrak{A}_1$ transforms all set functions in $\mathfrak{M}_{\Phi_1}$ into a null function, $\Phi_{2(U)}(E)$ is orthogonal to $\mathfrak{M}_{\Phi_1}$. Therefore, $\mathfrak{M}_{\Phi_2}$ is orthogonal to $\mathfrak{M}_{\Phi_1}$, and

$$\Phi_1^* \Phi_2(U, U') = (\Phi_{2(U')}, \Phi_{1(U)})_\beta = 0$$

for any sets U and U'. Hence

$$\begin{aligned}\mathfrak{A}\Phi_2(E, U) &= \mathfrak{A}\Phi_1\Phi_1^*\Phi_2(E, U) + \mathfrak{A}_1\Phi_2(E, U) \\ &= \mathfrak{A}_1\Phi_2(E, U).\end{aligned}$$

Therefore, from (3)

$$\int_U \lambda d_U \Phi_2(E, U) = \mathfrak{A}\Phi_2(E, U).$$

That is, $\Phi_2(E, U)$ is also a characteristic function of $\mathfrak{A}$.

Similarly,

$$\mathfrak{A}(E, E') - \mathfrak{A}\Phi_1 \Phi_1^*(E, E') - \mathfrak{A}\Phi_2 \Phi_2^*(E, E')$$

is a bounded self-adjoint kernel. Let $\Phi_3(E, U)$ be its characteristic function. Then, as proved above, $\Phi_3(E, U)$ is also a characteristic function of $\mathfrak{A}$, and $\mathfrak{M}_{\Phi_3}$ is orthogonal to $\mathfrak{M}_{\Phi_1}$ and $\mathfrak{M}_{\Phi_2}$.

By continuing this method, we get a system of characteristic functions of $\mathfrak{A}$

$$\Phi_1(E, U),\ \Phi_2(E, U),\ \ldots\ldots,\ \Phi_i(E, U),\ \ldots\ldots \tag{4}$$

so that any two of

$$\mathfrak{M}_{\Phi_1},\ \mathfrak{M}_{\Phi_2},\ \ldots\ldots,\ \mathfrak{M}_{\Phi_i},\ \ldots\ldots \tag{5}$$

are orthogonal. Such a system I will call *an orthogonal system of characteristic functions of* $\mathfrak{A}$. Since the space $\mathfrak{L}_2(\beta)$ is of denumerably infinite dimension, such an orthogonal system of characteristic functions is at most denumerably infinite. If there exists no characteristic function $\Phi(E, U)$, such that $\mathfrak{M}_\Phi$ is orthogonal to all the closed linear manifolds of (5), then I will say that (4) is *complete*.

Now, let (4) be a complete orthogonal system of characteristic functions of $\mathfrak{A}$. Then

$$\Phi_i(E, U)\,[=]_{E,U} \sum_\nu \psi_\nu^{(i)}(E)\overline{\eta_\nu^{(i)}(U)} \qquad (i = 1, 2, \ldots\ldots),$$

and

$$(\psi_\mu^{(i)}, \psi_\nu^{(j)})_\beta = 0$$

for any different values of i and j. Let $\left\{\psi_\nu^{(0)}(E)\right\}$ be a complementary normalized orthogonal system of $\left\{\psi_\nu^{(i)}(E)\right\}$ (ν, $i = 1, 2, \ldots\ldots$) in $\mathfrak{L}_2(\beta)$. Then $\mathfrak{A}(E, E')$ may be expressed as follows:

$$\mathfrak{A}(E, E')\,[=]_{E,E'} \sum_i \sum_\nu \phi_\nu^{(i)}(E)\overline{\psi_\nu^{(i)}(E')} + \sum_\nu \phi_\nu^{(0)}(E)\overline{\psi_\nu^{(0)}(E')},$$

where

$$\phi_\nu^{(i)}(E) = \mathfrak{A}\psi_\nu^{(i)}(E) \qquad \begin{pmatrix} i = 0, 1, 2, \ldots\ldots \\ \nu = 1, 2, 3, \ldots\ldots \end{pmatrix}.$$

Since

$$\mathfrak{A}\Phi_i\Phi_i^*(E, E')\,[=]_{E,E'} \sum_\nu \phi_\nu^{(i)}(E)\overline{\psi_\nu^{(i)}(E')},$$

$\sum_i \mathfrak{A}\,\Phi_i\,\Phi_i^*(E, E')$ converges strongly as functions of set E and of set E'. And

$$\mathfrak{A}(E, E') - \sum_i \mathfrak{A}\Phi_i\Phi_i^*(E, E')\,[=]_{E,E'} \sum_\nu \phi_\nu^{(0)}(E)\overline{\psi_\nu^{(0)}(E')}$$

is also a bounded self-adjoint kernel. Denote this kernel by $\mathfrak{A}_0(E, E')$. Then $\mathfrak{A}_0$ has at least a characteristic function, say $\Phi_0(E, U)$. Then, as before, $\mathfrak{M}_{\Phi_0}$ is orthogonal to all the closed linear manifolds of (5), and $\Phi_0(E, U)$ is also a characteristic function of $\mathfrak{A}$. But this contradicts the completeness of the system (4). Hence $\mathfrak{A}_0(E, E')$ must be identically zero.

Therefore, we have

$$\mathfrak{A}(E, E')\,[=]_{E,E'} \sum_i \mathfrak{A}\Phi_i\Phi_i^*(E, E).$$

But, since, as (2)

$$\mathfrak{A}\Phi_i\Phi_i^*(E, E') = \int_I \lambda D_{\sigma_i(U)}\Phi_i(E, \lambda)\overline{D_{\sigma_i(U)}\Phi_i(E', \lambda)}d\sigma_i(U) \qquad (i = 1, 2, \ldots\ldots),$$

we have

$$\mathfrak{A}(E, E')\,[=]_{E,E'} \sum_i \int_I \lambda D_{\sigma_i(U)}\Phi_i(E, \lambda)\overline{D_{\sigma_i(U)}\Phi_i(E', \lambda)}d\sigma_i(U). \quad (6)$$

(6) corresponds to the expansion of the kernel $\mathfrak{K}(E, E')$ of a completely continuous transformation with respect to the orthogonal system of characteristic functions:

$$\mathfrak{K}(E, E')\,[=]_{E, E'} \sum_i \lambda_i \Phi_i(E) \overline{\Phi_i(E')}\,, \tag{7}$$

where $\Phi_i(E)$ is a normalized characteristic function of $\mathfrak{K}(E, E')$ with respect to the characteristic constant λ_i.[1]

When $\mathfrak{A}(E, E')$ has only point spectrum, it is evident that (6) is of the form (7).

Properties of Characteristic Functions.

12. *Let* $\{\Phi_i(E, U)\}$ $(i = 1, 2, \ldots\ldots)$ *be an orthogonal system of characteristic functions of* $\mathfrak{A}$, *and* $\sigma_i(U)$ *be the bases corresponding to* $\Phi_i(E, U)$. *If* $\sum_i \sigma_i(U)$ *converges to a finite value* $\sigma(U)$ *for any Borel subsets* U *of* I, *then a characteristic function* $\Phi(E, U)$ *of* $\mathfrak{A}$ *with base* $\sigma(U)$ *exists, so that*

$$\Phi(E, U)\,[=]_{E, U} \sum_i \Phi_i(E, U) \qquad [\textit{in } \mathfrak{L}_2(\beta) \textit{ and } \mathfrak{L}_2(\sigma)].$$

As in the preceding section, we can express $\Phi_i(E, U)$ as follows:

$$\Phi_i(E, U)\,[=]_{E, U} \sum_\nu \psi_\nu^{(i)}(E)\overline{\eta_\nu^{(i)}(U)} \qquad [\text{in } \mathfrak{L}_2(\beta) \text{ and } \mathfrak{L}_2(\sigma_i)],$$

where
$$(\psi_\mu^{(i)}, \psi_\nu^{(j)})_\beta = 0$$

for any different values of i and j, and

$$\sum_\nu \eta_\nu^{(i)}(U)\overline{\eta_\nu^{(i)}(U')} = \sigma_i(UU')\,.$$

Since $\sigma_i(U)$ $(i = 1, 2, \ldots\ldots)$ are completely additive, non-negative set functions, it is evident that $\sigma(U)$ is likewise so. Now, $\{\psi_\nu^{(i)}(E)\}$ $(\nu,\ i = 1, 2, \ldots\ldots)$ is a normalized orthogonal system in $\mathfrak{L}_2(\beta)$ and

$$\sum_i \sum_\nu |\eta_\nu^{(i)}(U)|^2 = \sum_i \sigma_i(U)\,,$$

(1) Cf. F. Maeda, this journal, **3** (1933), 38, where the characteristic constant is defined as $\frac{1}{\lambda_i}$.

hence $\sum\limits_i \sum\limits_\nu \psi_\nu^{(i)}(E)\overline{\eta_\nu^{(i)}(U)}$ converges strongly as functions of set E. Therefore, a set function $\Phi(E, U)$ exists, so that

$$\Phi(E, U)\, [=]_E \sum_i \sum_\nu \psi_\nu^{(i)}(E)\overline{\eta_\nu^{(i)}(U)}\, [=]_E \sum_i \Phi_i(E, U) \quad [\text{in } \mathfrak{L}_2(\beta)] \tag{1}$$

for any set U. And

$$(\Phi_{(U)}, \Phi_{(U')})_\beta = \sum_i \sum_\nu \eta_\nu^{(i)}(U)\overline{\eta_\nu^{(i)}(U')} = \sum_i \sigma_i(UU') = \sigma(UU'). \tag{2}$$

It is evident that $\Phi_{(E)}(U)$ is additive, that is,

$$\Phi(E, U) = \Phi(E, U_1) + \Phi(E, U_2)$$

when $U = U_1 + U_2$. Now, let

$$U = U_1 + U_2 + \ldots\ldots + U_n + \ldots\ldots,$$

and put
$$M_n = U_1 + U_2 + \ldots\ldots + U_n .$$

Then
$$\Phi(E, U) - \Phi(E, M_n) = \Phi(E, U - M_n),$$

but, by (2)
$$\| \Phi_{(U-M_n)} \|_\beta^2 = \sigma(U - M_n).$$

Hence
$$\underset{n\to\infty}{[\lim]}\, \Phi_{(M_n)}(E) = \Phi_{(U)}(E),$$

therefore
$$\lim_{n\to\infty} \Phi(E, M_n) = \Phi(E, U).$$

That is, $\Phi_{(E)}(U)$ is completely additive.

Then, since (2) holds, by sec. 7, $\{\Phi_{(U)}(E)\}$ is a normalized orthogonal system with base $\sigma(U)$. But $\sum\limits_i \sum\limits_\nu \psi_\nu^{(i)}(E)\overline{\eta_\nu^{(i)}(U)}$ is nothing but the expansion of the bounded kernel $\Phi(E, U)$ with respect to $\{\psi_\nu^{(i)}(E)\}$ $(i, \nu = 1, 2, \ldots\ldots)$. Hence, by sec. 2,

$$\Phi(E, U)\, [=]_U \sum_i \sum_\nu \psi_\nu^{(i)}(E)\overline{\eta_\nu^{(i)}(U)}\, [=]_U \sum_i \Phi_i(E, U). \quad [\text{in } \mathfrak{L}_2(\sigma)].^{(1)} \tag{3}$$

---

(1) Generally, any set function $\zeta(U)$ in $\mathfrak{L}_2(\sigma_i)$ belongs to $\mathfrak{L}_2(\sigma)$. For $\|\zeta\|_{\sigma_i}^2 = \int |D_{\sigma_i(U)}^{(s)}\zeta(\lambda)|^2 d\sigma_i(U) = \int_I \left|\dfrac{D_{\sigma(U)}^{(s)}\zeta(\lambda)}{D_{\sigma(U)}^{(s)}\sigma_i(\lambda)}\right|^2 D_{\sigma(U)}^{(s)}\sigma_i(\lambda)d\sigma(U) \geqq \int_I |D_{\sigma(U)}^{(s)}\zeta(\lambda)|^2 d\sigma(U)$ $\|\zeta\|_\sigma^2$. And, since $\|\zeta_\nu - \zeta\|_{\sigma i} \geqq \|\zeta_\nu - \zeta\|_\sigma$, if $\{\zeta_\nu(U)\}$ converges strongly to (U) in $\mathfrak{L}_2(\sigma_i)$, then it will likewise converge strongly in $\mathfrak{L}_2(\sigma)$.

Consequently, by (1)

$$\sum_i \mathfrak{A}\Phi_i(E, U) = \mathfrak{A}\Phi(E, U) .$$

And by (3), $$\sum_i \int_U \lambda d_U \Phi_i(E, U) = \int_U \lambda d_U \Phi(E, U) .$$

But, since $$\int_U \lambda d_U \Phi_i(E, U) = \mathfrak{A}\Phi_i(E, U) \qquad (i = 1, 2, \ldots .),$$

we have $$\int_U \lambda d_U \Phi(E, U) = \mathfrak{A}\Phi(E, U) .$$

That is, $\Phi(E, U)$ is a characteristic function of $\mathfrak{A}$ with base $\sigma(U)$.

13. *Let $\Phi(E, U)$ be a characteristic function of $\mathfrak{A}$ with base $\sigma(U)$. If $\xi(U)$ be a set function in $\mathfrak{L}_2(\sigma)$, then*

$$\int_U D_{\sigma(U)}\Phi(E, \lambda) D_{\sigma(U)}\xi(\lambda) d\sigma(U) = \Psi(E, U) \tag{1}$$

is also a characteristic function of $\mathfrak{A}$ with base

$$\rho(U) = \int_U | D_{\sigma(U)}\xi(\lambda) |^2 d\sigma(U) .$$

Let, as sec. 10,

$$\Phi(E, U) [=]_{E, U} \sum_\nu \psi_\nu(E)\overline{\eta_\nu(U)} \qquad [\text{in } \mathfrak{L}_2(\beta) \text{ and } \mathfrak{L}_2(\sigma)],$$

then $$\Psi(E, U) = \sum_\nu \psi_\nu(E)\overline{\zeta_\nu(U)} ,$$

where $$\zeta_\nu(U) = \int_U D_{\sigma(U)}\eta_\nu(\lambda) \overline{D_{\sigma(U)}\xi(\lambda)} d\sigma(U) . \tag{2}$$

Then, by sec. 8, $\{\zeta_\nu(U)\}$ is a complete normalized orthogonal system in $\mathfrak{L}_2(\rho)$. Hence, $\{\Psi_{(U)}(E)\}$ is a normalized orthogonal system with base $\rho(U)$, and

$$\Psi(E, U) [=]_{E, U} \sum_\nu \psi_\nu(E)\overline{\zeta_\nu(U)} \qquad [\text{in } \mathfrak{L}_2(\beta) \text{ and } \mathfrak{L}_2(\rho)].$$

Since, $$\mathfrak{A}\Phi(E, U) [=]_{E, U} \sum_\nu \phi_\nu(E)\overline{\eta_\nu(U)} \qquad [\text{in } \mathfrak{L}_2(\beta) \text{ and } \mathfrak{L}_2(\sigma)]$$

where $$\phi_\nu(E) = \mathfrak{A}\psi_\nu(E) \qquad (\nu = 1, 2, \ldots .),$$

by (2), we have

$$\int_U D_{\sigma(U)}\mathfrak{A}\Phi(E,\lambda)D_{\sigma(U)}\xi(\lambda)d\sigma(U)=\sum_\nu \phi_\nu(E)\overline{\zeta_\nu(U)}$$
$$=\mathfrak{A}\Psi(E,U)\,. \tag{3}$$

On the other hand, by (1)

$$\int_U \lambda D_{\sigma(U)}\Psi(E,\lambda)d\sigma(U)=\int_U \lambda D_{\sigma(U)}\Phi(E,\lambda)D_{\sigma(U)}\xi(\lambda)d\sigma(U)\,. \tag{4}$$

Consequently, since

$$\int_U \lambda d_U\Phi(E,U)=\mathfrak{A}\Phi(E,U)$$

by (3) and (4) we have

$$\int_U \lambda d_U\Psi(E,U)=\mathfrak{A}\Psi(E,U)\,.$$

That is, $\Psi(E,U)$ is a characteristic function of $\mathfrak{A}$ with base $\rho(U)$.

14. Let $\Phi_i(E,U)$ and $\Phi_j(E,U)$ be two characteristic functions of $\mathfrak{A}$ with base $\sigma_i(U)$ and $\sigma_j(U)$ respectively. If $\mathfrak{M}_{\Phi_i}$ and $\mathfrak{M}_{\Phi_j}$ are orthogonal, as in sec. 11, we can express $\Phi_i(E,U)$ and $\Phi_j(E,U)$ as follows:

$$\Phi_i(E,U)\,[=]_{E,U}\sum_\nu \psi_\nu^{(i)}(E)\overline{\eta_\nu^{(i)}(U)}\qquad [\text{in } \mathfrak{L}_2(\beta) \text{ and } \mathfrak{L}_2(\sigma_i)],$$

$$\Phi_j(E,U)\,[=]_{E,U}\sum_\nu \psi_\nu^{(j)}(E)\overline{\eta_\nu^{(j)}(U)}\qquad [\text{in } \mathfrak{L}_2(\beta) \text{ and } \mathfrak{L}_2(\sigma_j)],$$

where
$$(\psi_\mu^{(i)},\psi_\nu^{(j)})_\beta=0\,. \tag{1}$$

Let $\xi_i(U)$ and $\xi_j(U)$ be two set functions in $\mathfrak{L}_2(\sigma_i)$ and $\mathfrak{L}_2(\sigma_j)$ respectively. Then, by the preceding section,

$$\Psi_i(E,U)=\int_U D_{\sigma_i(U)}\Phi_i(E,\lambda)D_{\sigma_i(U)}\xi_i(\lambda)d\sigma_i(U)$$
$$[=]_{E,U}\sum_\nu \psi_\nu^{(i)}(E)\overline{\zeta_\nu^{(i)}(U)}\qquad [\text{in } \mathfrak{L}_2(\beta) \text{ and } \mathfrak{L}_2(\rho_i)]$$

and
$$\Psi_j(E,U)=\int_U D_{\sigma_j(U)}\Phi_j(E,\lambda)D_{\sigma_j(U)}\xi_j(\lambda)d\sigma_j(U)$$
$$[=]_{E,U}\sum_\nu \psi_\nu^{(j)}(E)\overline{\zeta_\nu^{(j)}(U)}\qquad [\text{in } \mathfrak{L}_2(\beta) \text{ and } \mathfrak{L}_2(\rho_j)]$$

are also characteristic functions of $\mathfrak{A}$ with bases $\rho_i(U)$ and $\rho_j(U)$ respectively, where

$$\rho_i(U) = \int_U |D_{\sigma_i(U)}\xi_i(\lambda)|^2 d\sigma_i(U),$$

$$\rho_j(U) = \int_U |D_{\sigma_j(U)}\xi_j(\lambda)|^2 d\sigma_j(U).$$

Moreover, $\left\{\zeta_\nu^{(i)}(U)\right\}$, $\left\{\zeta_\nu^{(j)}(U)\right\}$ are complete normalized orthogonal systems in $\mathfrak{L}_2(\rho_i)$ and $\mathfrak{L}_2(\rho_j)$ respectively. Hence by (1) $\mathfrak{M}_{\Psi_i}$ and $\mathfrak{M}_{\Psi_j}$ are orthogonal.

Consequently, combining the theorems of sec. 12 and 13, we have the following theorem.

Let $\left\{\Phi_i(E, U)\right\}$ $(i = 1, 2, \ldots\ldots)$ *be an orthogonal system of characteristic functions of* $\mathfrak{A}$, *and* $\sigma_i(U)$ *be the bases corresponding to* $\Phi_i(E, U)$. *If* $\xi_i(U)$ *are set functions in* $\mathfrak{L}_2(\sigma_i)$, *so that*

$$\sum_i \int_U |D_{\sigma_i}(U)\xi_i(\lambda)|^2 d\sigma_i(U)$$

converges to a finite value, say $\sigma(U)$, *then*

$$\Phi(E, U) [=]_{E,U} \sum_i \int_U D_{\sigma_i(U)}\Phi_i(E, \lambda) D_{\sigma_i(U)}\xi_i(\lambda) d\sigma_i(U) \qquad [\textit{in } \mathfrak{L}_2(\beta) \textit{ and } \mathfrak{L}_2(\sigma)]$$

is a characteristic function of $\mathfrak{A}$ *with base* $\sigma(U)$.

15. Conversely, *when* $\left\{\Phi_i(E, U)\right\}$ *is a complete orthogonal system of characteristic functions of* $\mathfrak{A}$, *any characteristic function* $\Phi(E, U)$ *of* $\mathfrak{A}$ *with base* $\sigma(U)$, *is expressed as follows:*

$$\Phi(E, U) [=]_{E,U} \sum_i \int_U D_{\sigma_i(U)}\Phi_i(E, \lambda) D_{\sigma_i(U)}\xi_i(\lambda) d\sigma_i(U) \qquad [\textit{in } \mathfrak{L}_2(\beta) \textit{ and } \mathfrak{L}_2(\sigma)],$$

where $\xi_i(U)$ *are set functions in* $\mathfrak{L}_2(\sigma_i)$, $\sigma_i(U)$ *being the bases corresponding to* $\Phi_i(E, U)$.

As in sec. 11, let

$$\Phi_i(E, U)\,[=]_{E,\,U} \sum_\nu \psi_\nu^{(i)}(E)\overline{\eta_\nu^{(i)}(U)} \qquad [\text{in } \mathfrak{L}_2(\beta) \text{ and } \mathfrak{L}_2(\sigma_i)] \quad (1)$$

and $$(\psi_\mu^{(i)}, \psi_\nu^{(j)})_\beta = 0 \qquad \text{when} \quad i \neq j\,.$$

Let $\{\psi_\nu^{(0)}(E)\}$ be a complementary normalized orthogonal system of $\{\psi_\nu^{(i)}(E)\}$ ($\nu, i = 1, 2, \ldots\ldots$) in $\mathfrak{L}_2(\beta)$, and let $\Phi(E, U)$ be expanded with respect to $\{\psi_\nu^{(i)}(E)\}$ and $\{\psi_\nu^{(0)}(E)\}$, then

$$\Phi(E, U) = \Phi'(E, U) + \Phi''(E, U) \tag{2}$$

where $$\Phi'(E, U)\,[=]_{E,\,U} \sum_i \sum_\nu \psi_\nu^{(i)}(E)\overline{\zeta_\nu^{(i)}(U)} \qquad [\text{in } \mathfrak{L}_2(\beta) \text{ and } \mathfrak{L}_2(\sigma)],$$

$$\Phi''(E, U)\,[=]_{E,\,U} \sum_\nu \psi_\nu^{(0)}(E)\overline{\zeta_\nu^{(0)}(U)} \qquad [\quad ,, \quad].$$

Since, by sec. 9,

$$\begin{aligned}(\Phi_{(U')}, \Phi_{i(U)})_\beta &= (\Phi_{(UU')} + \Phi_{(U'-UU')}, \Phi_{i(UU')} + \Phi_{i(U-UU')})_\beta \\ &= (\Phi_{(UU')}, \Phi_{i(UU')})_\beta\,,\end{aligned}$$

$(\Phi_{(U')}, \Phi_{i(U)})_\beta$ is a function of UU', say $\xi_i(UU')$. On the other hand, from (1) and (2)

$$(\Phi_{(U')}, \Phi_{i(U)})_\beta\,[=]_U \sum_\nu \overline{\zeta_\nu^{(i)}(U')}\eta_\nu^{(i)}(U) \qquad [\text{in } \mathfrak{L}_2(\sigma_i)]\,.$$

But, $\{\eta_\nu^{(i)}(U)\}$ ($\nu = 1, 2, \ldots\ldots$) is a normalized orthogonal system in $\mathfrak{L}_2(\sigma_i)$, $\xi_i(UU')$, as a function of U, belongs to $\mathfrak{L}_2(\sigma_i)$. Hence $\xi_i(U) = \xi_i(UI)$ is a set function in $\mathfrak{L}_2(\sigma_i)$.

Put $$\mathfrak{P}_i(U, U') = \xi_i(UU')$$

then $$\begin{aligned} D_{\sigma_{i(U)}}\mathfrak{P}_{i(U')}(\lambda) &= D_{\sigma_{i(U)}}\xi_i(\lambda) && \text{when } \lambda \text{ is a point of } U', \\ &= 0 && \text{when } \lambda \text{ is not a point of } U'. \end{aligned}$$

Since $$\mathfrak{P}_i(U, U')\,[=]_U \sum_\nu \overline{\zeta_\nu^{(i)}(U')}\eta_\nu^{(i)}(U) \qquad [\text{in } \mathfrak{L}_2(\sigma_i)]$$

is an expansion of $\mathfrak{P}_{i(U')}(U)$ with respect to $\left\{\eta_\nu^{(i)}(U)\right\}$ $(\nu=1, 2, \ldots\ldots)$, we have

$$\zeta_\nu^{(i)}(U') = (\mathfrak{P}_{i(U')},\ \eta_\nu^{(i)})_{\sigma_i} = \int_I D_{\sigma_{i(U)}}\mathfrak{P}_{i(U')}(\lambda)\overline{D_{\sigma_{i(U)}}\eta_\nu^{(i)}(\lambda)}d\sigma_i(U)$$
$$= \int_{U'} D_{\sigma_{i(U)}}\xi_i(\lambda)\overline{D_{\sigma_{i(U)}}\eta_\nu^{(i)}(\lambda)}d\sigma_i(U)\,, \tag{3}$$

and

$$\sum_\nu |\,\zeta_\nu^{(i)}(U')\,|^2 = \|\,\mathfrak{P}_{i(U')}\,\|^2_{\sigma_i} = \int_I |\,D_{\sigma_{i(U)}}\mathfrak{P}_{i(U')}(\lambda)\,|^2\,d\sigma_i(U)$$
$$= \int_{U'} |\,D_{\sigma_{i(U)}}\xi_i(\lambda)\,|^2\,d\sigma_i(U)\,.$$

Since, by (2),

$$\|\,\Phi_{(U)}\,\|^2_\beta = \sum_i\sum_\nu |\,\zeta_\nu^{(i)}(U)\,|^2 + \sum_\nu |\,\zeta_\nu^{(0)}(U)\,|^2,$$

$\sum_i \int_U |\,D_{\sigma_{i(U)}}\,\xi_i(\lambda)|^2 d\sigma_i(U) = \sum_i\sum_\nu |\,\zeta_\nu^{(i)}(U)\,|^2$ converges. Hence, by sec. 14,

$$\sum_i \int_U D_{\sigma_{i(U)}}\Phi_i(E,\lambda)D_{\sigma_{i(U)}}\xi_i(\lambda)d\sigma_i(U) \tag{4}$$

is a characteristic function of $\mathfrak{A}$. But, by (1) and (3), (4) is equal to

$$\sum_i\sum_\nu \psi_\nu^{(i)}(E)\int_U \overline{D_{\sigma_{i(U)}}\eta_\nu^{(i)}(\lambda)}\ D_{\sigma_{i(U)}}\,\xi_i(\lambda)d\sigma_i(U)$$
$$= \sum_i\sum_\nu \psi_\nu^{(i)}(E)\,\zeta_\nu^{(i)}(U) = \Phi'(E, U)\,.$$

Hence $\qquad \int_U \lambda\, d_U\Phi'(E, U) = \mathfrak{A}\Phi'(E, U)\,.$

But, by the assumption,

$$\int_U \lambda\, d_U\Phi(E, U) = \mathfrak{A}\Phi(E, U)\,.$$

Hence, we have

$$\int_U \lambda\, d_U\Phi''(E, U) = \mathfrak{A}\Phi''(E, U)\,.$$

Therefore, $\Phi''(E, U)$ is a characteristic function of $\mathfrak{A}$, and $\mathfrak{M}\Phi''$ is orthogonal to all $\mathfrak{M}\Phi_i$ $(i = 1, 2, \ldots\ldots)$. But, since $\left\{ \Phi_i(E, U) \right\}$ $(i = 1, 2, \ldots\ldots)$ is a complete system of characteristic functions of $\mathfrak{A}$, $\Phi''(E, U)$ must be identically zero. Consequently,

$$\Phi(E, U) = \Phi'(E, U) \, [=]_{E, U} \sum_i \int_U D_{\sigma_i(U)} \Phi_i(E, \lambda) D_{\sigma_i(U)} \xi_i(\lambda) d\sigma_i(U)$$

$$[\text{in } \mathfrak{L}_2(\beta) \text{ and } \mathfrak{L}_2(\sigma)].$$