An Application of Certain Geometrical Transformation, Especially on Poristic Theorems.

By

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J. V. Poncelet has proved the following elegant theorem in his celebrated work.[1]

If there be a polygon inscribed in a conic K_1 and circumscribed to another conic K_2, then an infinite number of such polygons exists, inscribed in K_1 and circumscribed to K_2 (called Poncelet's polygons); or if a Poncelet's polygon be constructed, then any polygonal configuration inscribed in K_1 and circumscribed to K_2 and starting from any point will always be closed: and conversely, if a polygonal series of points inscribed in K_1 and circumscribed to K_2 does not close, then every other such polygonal configuration will never be closed, whereever it may start from. (Fig. 1)

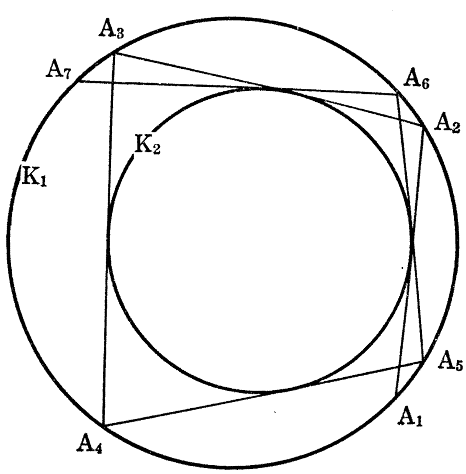

Fig. 1.

(1) *Traité der ropriétés projectives des Fugyres*, (1833), 361.

The above porism of Poncelet has been since discussed by several writers and generalised in various directions.(1)

J. Steiner has demonstrated another porism relating to any circle-ring :(2)

In the space between two given circles K_1 and K_2, of which K_2 lies wholly within K_1, there is a series of circles X_1, X_2, ... with centres O_1, O_2,...., all of which are tangential to K_1 and K_2 and also to their neighbours in the series. If, now, X_{m+1} coincide with X_1 traversing the space n times (thus we obtain the so-called Steinerian series of circles), then an infinite number of Steinerian series will exist starting from any circle which touches both K_1 and K_2, and the series consists of m circles traversing the space n times. (Fig. 2)

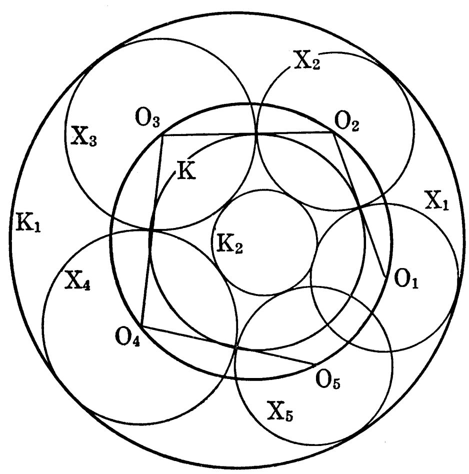

Fig. 2.

As these two porisms are analogous, we may expect an intrinsic connection between them. P. Serret,(3) being interested in this line, observed that the centres O_1, O_2, ... lie on a fixed conic and that the segments O_1O_2, O_2O_3, ... envelop a fixed circle which is coaxial with K_1 and K_2 cutting X_1, X_2, ... orthogonally. (Fig. 2). J. Larmor(4) happend to discover the fact independently. A. Hurwitz(5)

(1) *Encykl. d. Math. Miss.* Bd. III, 2. Teil, 1. Hälft, 46.
(2) *J. für Math.* **1**, (1826), 256.
(3) *Nouv, Annal.* (2) **1**, (1862), 184.
(4) *Messenger of Math.* **13**, (1884), 61.
(5) *Math. Ann.* **15**, (1878), 16.

has made clear the relation from an algebraic point of view, that is, he has demonstrated the porisms standing on the theorem that if a problem whose solutions are to be determined through roots of an equation of the m^{th} degree, allows more than m solutions, then an infinite number of solutions exists. A Emch[1] has derived a generalisation of Steiner's porism from a study of linkage in certain cases, and then, inverting a special case in a different direction, obtained also Poncelet's porism from this general theorem.

Now, the present writer has tried to get the direct relation which connects these porisms by the use of a certain geometrical transformation which is well-known in the circle geometry of Möbius, since these theorems can be enunciated in terms of circle geometry.

If we take tetracyclie co-ordinates of circles in a plane as being point co-ordinates in tridimensional space S, obtain a hyperbolic geometry with an absolute sphere A. The correspondence between the spaces are tabled as follows :[2]

In plane π:	**In hyperbolic Space S with Absolute A:**
Circle	Point
Null circle	Point in A
Angle of two not null circles . .	Distance of two points not on A
Mutually orthogonal circles . . .	Points conjugate with respect to A
Caoxial system of circles	Line
Two circles tangent to each other	Two points whose connector is tangent to A
Three circles which have a common point	Three points whose plane is tangent to A
Circles mutually inverse in proper circle, or reflexions of one another in a non isotropic line .	Reflexions in a point not on A, or reflexions of one another in a point on a definite tangent plane to A

The above correspondence could also be obtained geometrically by reciprocating the result with respect to A after having projected the original figure in π stereographically on A[3].

(1) *Annals of Math.* **1**, No. 2.

(2) These are well-known, see for example, Coolidge, A Treatise on the Circle and the Sphere, 133.

(3) See, for example, Blaschke, Vorlesungen ü. Diff-geo. Bd. III, ss 7-9.

By means of this table, we can transform several theorems and hence a few examples are presented below, especially bearing on the poristic theorems mentioned above.

The correspondent to Steiner's porism:

There is a series of points $X_1, X_2, \ldots, X_m, \ldots$ *all of which are tangent*(1) *to two given points* K_1 *and* K_2 *and also to their neighbours in the series. If, now,* X_{m+1} *coincide with* X_1*, the system of points travers-*

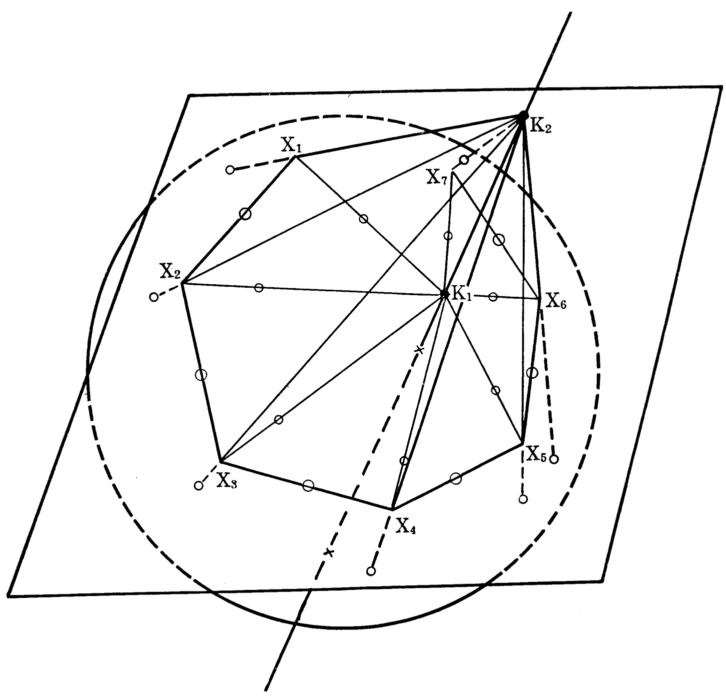

Fig. 3.

ing n times around A, then an infinite number of such polygons exists starting from any point which touches both K_1 *and* K_2*, and the polygons consist of m points traversing n times around A.* (Fig. 3)

Since all the points which touch K_1 and K_2 are on two planes Σ_1 and Σ_2, bisectors of the planes K_1 and K_2,(2) the locus of these points

(1) The term "a point A is tangent to a point B" is here used in the sense that the line AB touches the Absolute A.

(2) Plane K means the polar plane of point K with respect to A.

becomes two conics, one on Σ_1 and the other on Σ_2, which locus is the intersection of two cones K_1 and K_2.[1] Cutting the present configuration with say Σ_1, we obtain :

A polygonal set of points X_1, X_2, ... *is inscribed in a conic* M_1 *and circumscribed to a circle* M_2. *If this close with m points traversing n times around* M_2, *then an infinite number of such polygons exists starting from any point on* M_1.

In essentials, this is nothing but Poncelet's porism.

Next, the present writer submits a generalisation of Steiner's porism, the proof of which, being very simple, is here omitted for the sake of briefness:

A series of circles X_1, X_2, ... *each of which touches their neighbours in the series are so situated that each of them is tangent to some pair of circles* K_i, K_i' *of another system of coaxial circles K;* K_1, K_1'; K_2, K_2'; .... *where* K_i *and* K_i' *are inverse with respect to K. If this series close with m circles, then an infinite number of such series will exist starting from any circle which touches* K_1 *and* K_1'.

Corresponding to the above, we have

A series of points X_1, X_2 .... *each of which touches its neighbours in the series, is so arranged that each point is tangent to a pair of points* K_i, K_i', *in the other range of points K;* K_1, K_1' ; K_2, K_2' ; .... *where* K_i, K_i' *are equidistant from K. If the series close with m points, then there is an infinite number of such series starting from any desired points which touch* K_1 *and* K_1'. (Fig. 4)

Deduction from the above :

Since, as in the preceding case, the points X_1, X_2, ... may be conceived to be on a fixed plane, we obtain the following enunciation by cutting the configuration with this plane :

If a polygon X_1, X_2, ... *is circumscribed to a circle M and has its vertices* X_1, X_2, ... *lying on the respective given conics* K_1, K_2, ...[2], *then an infinite number of such polygons exists starting from any point on the first conic* K_1.

Another generalisation of Steiner's porism by A. Emch which states that,

(1) Cone K means the cone which envelops A, and has its vertex at K.

(2) Conic K is the conic obtained as the section of the cone K.

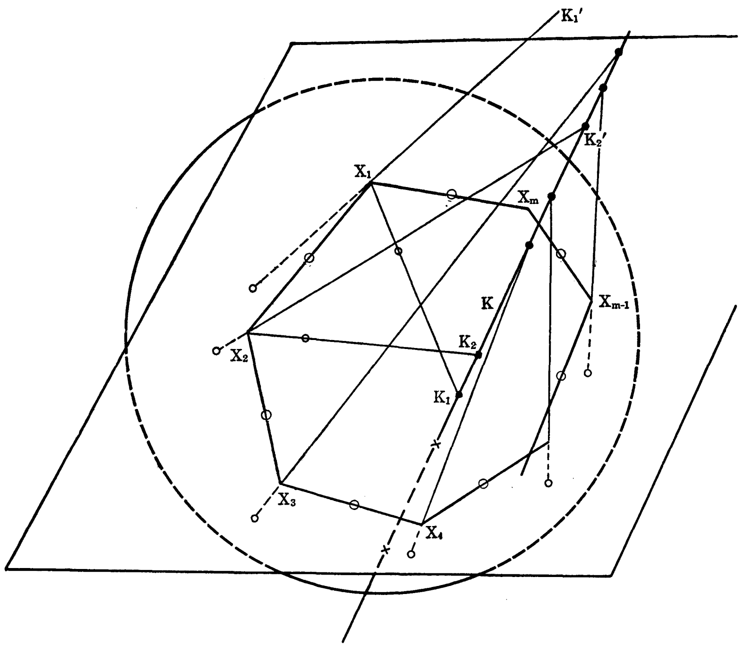

Fig. 4.

There is a series of circles X_1, X_2, ... *each of which touches two fixed circles* K_1 *and* K_2, *and each pair of consecutive circles* X_i, X_{i+1} *in the series intersects in two points* B_i, B_i' *of which* B_1, B_2, ... *are on a circle* K_3 *(then necessarily the points* B_1', B_2', ... *are also situated on another circle* K_4 *which is inverse with* K_3 *with respect to the circle that cuts* X_i *orthogonally). If* B_{m+1} *coincide with* B_1, *that is, if this series of circles be closed, then there is an infinite number of such closed set based upon the three circles* K_1, K_2 *and* K_3 *containing m circles,*

Offers the correspondent that:

There is a series of points X_1, X_2, ... *each of which touches two given points* K_1, K_2 *and each plane determined by two consecutive points in the series with a third fixed point* K_3 *is always tangent to* A. *If* X_{m+1} *coincide with* X_1, *that is, if the series be closed, then an infinite number of such closed series of m points exist, based upon the three given points* K_1, K_2 *and* K_3. (Fig. 5)

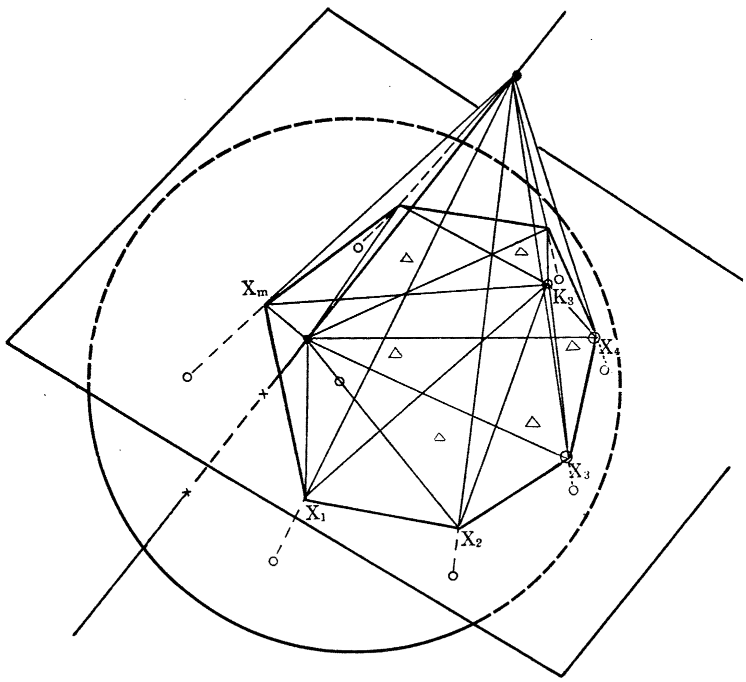

Fig. 5.

If we take a section of the configuration of Fig. 5 with a suitable plane, we obtain Poncelet's porism; but conversely it would be more interesting to use Poncelet's theorem in this plane in order to prove Emch's generalisation.

A third generalisation, whose enunciation is here omitted, is also possible by combining the two generalisations made above. The correspondent should be used in the present case also as a proof of the original proposition, since the deduction does not differ in essentials from those of the first generalisation.

Correspondent:

A series of points X_1, X_2, X_3 .... *are so situated that each of them is tangent to some pair* K_i, K_i' *of a range of points* K; K_1, K_1'; K_2, K_2'; .... *where* K_i *and* K_i' *are equidistant from* K, *and each plane, determined by two consecutive points in the series with a given point* $\bar{K}$ *is tangent to* A. *If* X_{m+1} *is coincident with* X_1, *then there is an infinite number of*

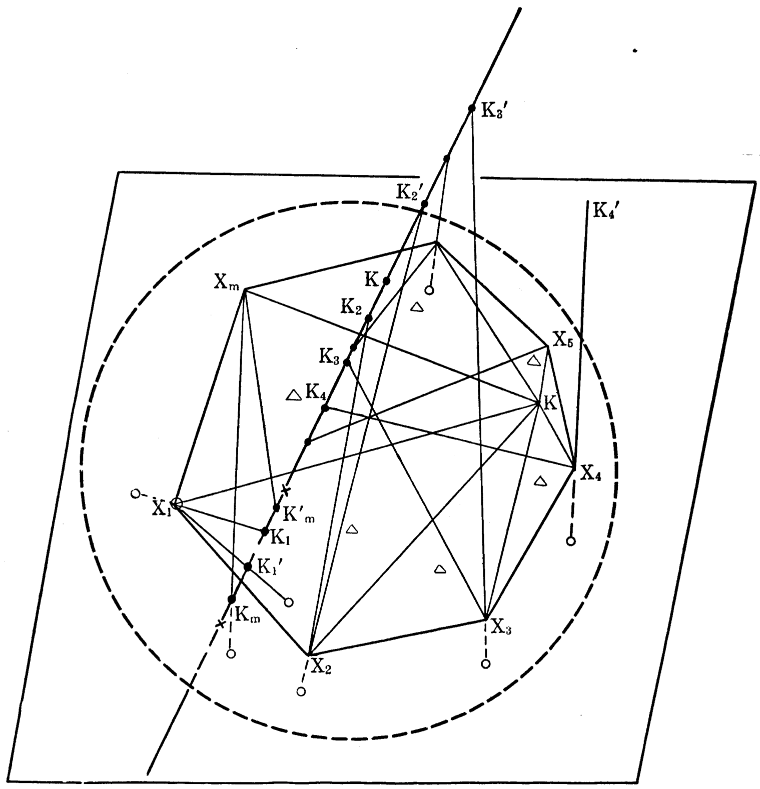

Fig. 6.

such closed set of m points based on $\bar{K}$ and the given range of points. (Fig. 6)

Further examples should be treated, for Poncelet's porism relating to coaxial system of circles e.g.,

Correspondent:

A range of points K, K_1, K_2, ... being given, there is a set of point X_1, X_2, ... on a tangent plane to A such that each of them touches the respective point of the given range, and further the planes determined by any two consecutive points in the series and K are tangent to A. If X_{m+1} coincide with X_1, then an indefinite number of such closed sets of points will exist.

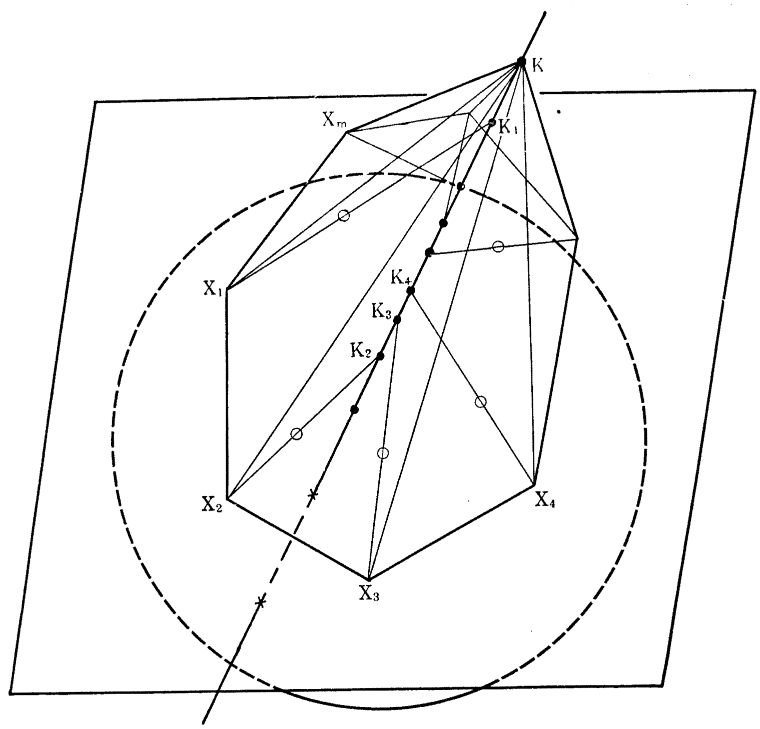

Fig. 7.

Cutting the above configuration by the tangent plane on which X_i lie and transforming the result by suitable collineation, we obtain:

If a polygon can be circumscribed to a conic of a confocal system of conics so that its vertices lie on the respective conics of the system, then there is an infinite number of such polygons.

It is worthy of note that the result can be obtained directly as the reciprocal theorem of Poncelet's Porism.

As was seen above, this transformation offers considerable facilities for deriving various theorems from theorems proved in the circle geometry, and conversely, theorems of the latter can, in some instances, be derived from that of the former. Actually this reciprocal procedure permits certain new generalisation of Poncelet's porism. But it seems preferable, instead of cumbersome enunciations of these new generalisations, to state that Mr. K. Ogura's non-poristic theorem.[1]

"*The circles which cut the opposite sides of circular inscribed hexagon and the circumcircle are concurrent*" can also by using the above transformation, be proved to be the same as that of Brianchon relating to a bundle of planes.

(1) Tôkyô Su. But. Kw. K. [2] **6** (1911–1912) 159.