On the General Derivatives of the Set Functions

By

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Several attempts[1] have been made to obtain the fundamental relation between the generalised integrals and their derivatives with respect to any set function $\varphi(E)$, which is similar to the relation between the Lebesgue integrals and their derivatives. But all of them are only for the special derivatives and not for the general derivatives as in the case of the Lebesgue measure. The difficulties besetting such attempts arise from the fact that the set function $\varphi(E)$ in general has not the following characteristic properties of the Lebesgue measure:

1. All the congruent measurable sets have the same Lebesgue measure.

2. The ratio of the Lebesgue measures of any two concentric intervals whose edges are in a fixed ratio is constant.

To keep clear of these difficulties, I define the uniform monotonity and the poles of $\varphi(E)$ and I prove that except the set of poles the same relations hold as in the case of the Lebesgue measure, using Vitali's theorem and net derivatives. And I find that when $\varphi(E)$ is absolutely continuous with respect to the Lebesgue measure, there exists no such exception, since the set of poles have the φ-value zero.

Uniform Monotonity of the Set Function

1. Let $\varphi(E)$ be a completely additive set function defined in a closed family of sets containing all the Borel sets in the metric space R which is compact in itself. Then we say, with de la Vallée Poussin,[2] that a set A of the family is *normal* (φ) if, for any prescribed positive number ϵ, we can find an open set $O \supset A$ and a closed set $F \subset A$ so that

(1) For the references, cf. A. J. Maria, *Annales of Mathematics* II, **28** (1927), 419; and P. J. Daniell, *Proc. London Math. Soc.*, (2), **30** (1929), 187.

(2) De la Vallée Poussin, *Intégrales de Lebesgue*, (1916), 86.

$$|\varphi(A'-A'')|<\epsilon$$

whenever $O>A'>A''>F$, where the sets A', A'' belong to the above family.

If we denote by $\mathfrak{F}$ a closed family of φ-normal sets which contains all the Borel sets, then $\varphi(E)$ is also completely additive in $\mathfrak{F}$, and there exist for $\varphi(E)$ the total variation $\overline{\varphi}(E)$, the positive variation $\varphi^+(E)$ and the negative variation $\varphi^-(E)$. These variations are completely additive in $\mathfrak{F}$ and they have the following relations:

$$\varphi(E)=\varphi^+(E)-\varphi^-(E)\ ,$$

$$\overline{\varphi}(E)=\varphi^+(E)+\varphi^-(E)\ .$$

Then any set A of $\mathfrak{F}$ can be decomposed into two sets A^+ and A^- of $\mathfrak{F}$ so that

$$A=A^++A^-\ ,$$

and

$$\varphi^+(A^+)=\varphi(A^+)\ , \qquad \varphi^-(A^+)=0\ ,$$

$$\varphi^-(A^-)=-\varphi(A^-)\ , \qquad \varphi^+(A^-)=0\ .^{(1)}$$

I call for brevity A^+ and A^-, *the positive and negative parts of A respectively with respects to $\varphi(E)$.*

Let λ be a positive number greater than 1, and a be a point in the space R. If there exist positive numbers $\kappa(a, \lambda)$ and ρ_λ so that for any positive number $\rho<\rho_\lambda$

$$\overline{\varphi}\left\{\overline{U}(a, \rho)\right\}\geqq\kappa(a, \lambda)\overline{\varphi}\left\{\overline{U}(a, \lambda\rho)\right\}\ , \tag{1}$$

where $\overline{U}(a, \rho)$ means a closed neighbourhood of a with radius ρ, then I say that the function $\varphi(E)$ is *uniformly monotone at the point a.* If at a point a and a value ρ', $\overline{\varphi}\left\{\overline{U}(a, \rho')\right\}=0$, then $\overline{\varphi}\left\{\overline{U}(a, \rho)\right\}=0$ for all values of $\rho\leqq\rho'$. In such a case I say that the point a is a *zero point* of $\varphi(E)$. At such a point taking $\rho_\lambda=\dfrac{\rho'}{\lambda}$, the condition (1) is always satisfied for $\rho<\rho_\lambda$. Therefore, at the zero points $\varphi(E)$ is always uniformly monotone. If the point a is not a zero point, consider the limit

(1) H. Hahn, *Theorie der reellen Funktionen*, (1921), 404.

$$\varliminf_{\rho\to 0}\frac{\overline{\varphi}\{\overline{U}(a,\rho)\}}{\overline{\varphi}\{\overline{U}(a,\lambda\rho)\}}=l(a,\lambda)\ .$$

If $l(a,\lambda)>0$, then $\varphi(E)$ is uniformly monotone at a, and $\kappa(a,\lambda)$ may take any positive value which is less than $l(a,\lambda)$. If $l(a,\lambda)=0$, $\varphi(E)$ is not uniformly monotone at a. In this case I say that the point a is a *pole* of $\varphi(E)$.

The above definition of the uniform monotonity of $\varphi(E)$ is irrelevant to the values of λ; that is, if $\varphi(E)$ is uniformly monotone at a with respect to a value λ then it is so also with respect to another value λ'. For if $\lambda'<\lambda$, we have only to take $\kappa(a,\lambda')=\kappa(a,\lambda)$ and $\rho_{\lambda'}=\rho_\lambda$. Next if $\lambda'>\lambda$, then there is a positive integar n so that $\lambda^n>\lambda'$. Let $\rho_{\lambda'}=\frac{\rho_\lambda}{\lambda^{n-1}}$, then putting $\lambda^i\rho$ for ρ in (1), for any $\rho<\rho_{\lambda'}$ we have following n inequalities

$$\overline{\varphi}\left\{\overline{U}(a,\lambda^i\rho)\right\}\geqq\kappa(a,\lambda)\overline{\varphi}\left\{\overline{U}(a,\lambda^{i+1}\rho)\right\}\qquad(i=0,1,2,\ldots,n-1).$$

From these inequalities we have by multiplication

$$\overline{\varphi}\left\{\overline{U}(a,\rho)\right\}\geqq\left[\kappa(a,\lambda)\right]^n\overline{\varphi}\left\{\overline{U}(a,\lambda^n\rho)\right\}\ .$$

Since $\lambda^n>\lambda'$, for any $\rho<\rho_{\lambda'}$ we have

$$\overline{\varphi}\left\{\overline{U}(a,\rho)\right\}\geqq\left[\kappa(a,\lambda)\right]^n\overline{\varphi}\left\{\overline{U}(a,\lambda'\rho)\right\}\ .$$

Hence take $\kappa(a,\lambda')=\left[\kappa(a,\lambda)\right]^n$ then (1) is valid for λ', that is, $\varphi(E)$ is uniformly monotone at a with respect to λ'.

As to the pole of $\varphi(E)$, it is also irrelevant to the values of λ. For, let a point a be a pole of $\varphi(E)$ with respect to a value λ. If $\varphi(E)$ is uniformly monotone at a with respect to another value λ', then by the above it is so also with respect to λ. This is absurd. Therefore, the point a is also the pole of $\varphi(E)$ with respect to λ'.

Let A be a set in the space R. Then if there exist positive numbers $\kappa(A,\lambda)$ and ρ_λ so that at any point a of A

$$\overline{\varphi}\left\{\overline{U}(a,\rho)\right\}\geqq\kappa(A,\lambda)\overline{\varphi}\left\{\overline{U}(a,\lambda\rho)\right\}$$

for any $\rho < \rho_\lambda$, then we say that the function $\varphi(E)$ is *uniformly monotone at the set* A. As we have seen, when $\varphi(E)$ is uniformly monotone at A with respect to a value of λ, then it is so also for any value of λ which is greater than 1.

2. *Let A be any set belonging to $\mathfrak{F}$. If we denote the set of zero points of $\varphi(E)$ in A by A_z, then A_z also belongs to $\mathfrak{F}$ and $\bar{\varphi}(A_z)=0$.*

Now for any point z of A_z there corresponds an open neighbourhood $U(z, \rho)$ so that $\bar{\varphi}\{U(z, \rho)\}=0$; from all such neighbourhoods, since R is separable, we can select an infinite sequence

$$U(z_1, \rho_1), \quad U(z_2, \rho_2), \quad \ldots\ldots, \quad U(z_n, \rho_n), \quad \ldots\ldots,$$

so that each point of A_z is interior to at least one of them,[1] that is

$$A_z \subseteqq U(z_1, \rho_1) \dot{+} U(z_2, \rho_2) \dot{+} \ldots\ldots \dot{+} U(z_n, \rho_n) + \ldots\ldots .$$

But as it is obvious that all the points of $A\{U(z_1, \rho_1) \dot{+} U(z_2, \rho_2) \dot{+} \ldots\}$ are zero points of $\varphi(E)$ in A, we have

$$A_z = A\{U(z_1, \rho_1) \dot{+} U(z_2, \rho_2) \dot{+} \ldots\ldots \dot{+} U(z_n, \rho_n) \dot{+} \ldots\ldots\} .$$

Therefore, A_z belongs to $\mathfrak{F}$. Moreover $\bar{\varphi}(A_z)=0$, since

$$\bar{\varphi}\{U(z_1, \rho_1)\} + \bar{\varphi}\{U(z_2, \rho_2)\} + \ldots\ldots + \bar{\varphi}\{U(z_n, \rho_n)\} + \ldots\ldots = 0 .$$

The point function $l(a, \lambda)$ is of Baire's class.

To prove this, consider $\bar{\varphi}\{\bar{U}(a, \rho)\}$ as a function of the point a, ρ being constant. Let a' be a point so that $r(a, a') < \delta$, where δ is any given positive number. Consider two sets V and V' so that

$$\bar{U}(a, \rho) = \bar{U}(a, \rho)\bar{U}(a', \rho) + V ,$$

$$\bar{U}(a', \rho) = \bar{U}(a, \rho)\bar{U}(a', \rho) + V' ;$$

then $$\left|\bar{\varphi}\{\bar{U}(a, \rho)\} - \bar{\varphi}\{\bar{U}(a' \rho)\}\right| = \left|\bar{\varphi}(V) - \bar{\varphi}(V')\right| < \bar{\varphi}(V) + \bar{\varphi}(V') .$$

But since $$\bar{U}(a, \rho) < U(a', \rho + \delta),$$

(1) H. Hahn, *loc. cit.*, 91.

we have $\qquad V < U(a', \rho+\delta) - \overline{U}(a', \rho)$.

Since $\qquad \lim_{\delta\to 0} U(a', \rho+\delta) = \overline{U}(a', \rho)$,

we have $\qquad \lim_{\delta\to 0} \overline{\varphi}(V) = 0$.

Similarly $\qquad \lim_{\delta\to 0} \overline{\varphi}(V') = 0$.

Therefore $\qquad \lim_{\delta\to 0} \left| \overline{\varphi}\left\{\overline{U}(a, \rho)\right\} - \overline{\varphi}\left\{\overline{U}(a', \rho)\right\} \right| = 0$,

that is, $\overline{\varphi}\left\{\overline{U}(a, \rho)\right\}$ is continuous at the point a. Therefore, $\dfrac{\overline{\varphi}\{\overline{U}(a, \rho)\}}{\overline{\varphi}\{\overline{U}(a, \lambda\rho)\}}$, hence also $l(a, \lambda)$ are Baire's functions.

A belonging to $\mathfrak{F}$, the set $A\left\{l(a, \lambda) > \alpha\right\}$ of the points a of A, α being a positive number less than 1, belongs also to $\mathfrak{F}$, since this set is a product of A and a Borel set. Hence the set $A\left\{l(a, \lambda) > 0\right\}$ belongs also to $\mathfrak{F}$. Therefore, *the set A_p of the poles in A belongs to* $\mathfrak{F}$, since

$$A_p = A - A\left\{l(a, \lambda) > 0\right\} - A_z .$$

3. If the metric space R is an Euclidian space R_n of n dimensions, we may use in the definitions of the uniform monotonity and poles of $\varphi(E)$, the closed intervals $\overline{W}(a, \rho)$ instead of the closed neighbourhoods $\overline{U}(a, \rho)$, where $\overline{W}(a, \rho)$ means the set of points of coordinates $(x_1, x_2, \ldots\ldots, x_n)$ satisfying

$$a_i - \frac{\rho}{2} \leqq x_i \leqq a_i + \frac{\rho}{2} \qquad (i=1, 2, \ldots\ldots, n) ,$$

and $(a_1, a_2, \ldots\ldots, a_n)$ are the coordinates of the point a.

For, if $$\overline{\varphi}\left\{\overline{U}(a, \rho)\right\} \geqq \kappa(a, \lambda)\, \overline{\varphi}\left\{\overline{U}(a, \lambda\rho)\right\} \tag{1}$$

for any $\rho < \rho_\lambda$, then since

$$\overline{W}(a, 2\rho) \geqq \overline{U}(a, \rho) \geqq \overline{W}\left(a, 2\frac{\rho}{\sqrt{n}}\right) ,$$

we have $$\overline{\varphi}\left\{\overline{W}(a, 2\rho)\right\} \geqq \kappa(a, \lambda)\overline{\varphi}\left\{\overline{W}\left(a, 2\lambda\frac{\rho}{\sqrt{n}}\right)\right\}$$

for any $\rho < \rho_\lambda$. Therefore, we have

$$\bar{\varphi}\{\overline{W}(a, \rho)\} \geqq \kappa'(a, \lambda')\bar{\varphi}\{\overline{W}(a, \lambda'\rho)\} \tag{2}$$

for any $\rho < 2\rho_\lambda$, where $\lambda' = \frac{\lambda}{\sqrt{n}}$ and $\kappa'(a, \lambda') = \kappa(a, \lambda)$. Similarly we can deduce (1) from (2).

General Derivatives in the Metric Space

4. Under the condition of the uniform monotonity we shall prove the following *Vitali's theorem* with respect to $\varphi(E)$:

Let $\varphi(E)$ be a completely additive, non-negative, function of normal sets, and A be a φ-normal set where $\varphi(A) \neq 0$, and $\varphi(E)$ be uniformly monotone at A. If at any point a of A there is a sequence of enumerably infinite closed neighbourhoods $\overline{U}(a, \rho)$, which converges to the point a, then from these we can select an enumerable, or a finite, set of neighbourhoods

$$\overline{U}(a_1, \rho_1), \quad \overline{U}(a_2, \rho_2), \quad \ldots\ldots, \quad \overline{U}(a_n, \rho_n), \quad \ldots\ldots, \tag{1}$$

so that

1. *any two of these have no common points, and*
2. *if we denote the sum of* (1) *by S, then*

$$\varphi(A - AS) = 0 \qquad \textit{and} \qquad \varphi(S - AS) < \epsilon$$

for any given positive number ϵ.

Since $\varphi(E)$ is uniformly monotone at A, for any point a of A we have

$$\varphi\{\overline{U}(a, \rho)\} \geqq \kappa(A, 3)\varphi\{\overline{U}(a, 3\rho)\} \tag{2}$$

for any $\rho < \rho'$.

A being normal (φ) and $\varphi(A) > 0$, there exists a closed set F' contained in A, so that $\varphi(F') > 0$. Let ϵ' be a positive number smaller than either ϵ or $\frac{\kappa(A, 3)\varphi(F')}{2}$, then we can find an open set O_1 and a closed set F so that $O_1 \supseteq A \supseteq F$ and

$$\varphi(O_1)-\varphi(F)<\epsilon' . \tag{3}$$

Let $F_1 = F \dot{+} F'$. Then, since

$$O_1 \geqq A \geqq F_1 \geqq F ,$$

$$\varphi(O_1)-\varphi(F_1)<\frac{\kappa(A, 3)\varphi(F')}{2} \leqq \frac{\kappa(A, 3)\varphi(F_1)}{2} ,$$

we have $$\varphi(O_1)<\left(1+\frac{\kappa(A, 3)}{2}\right)\varphi(F_1) . \tag{4}$$

For any point a of F_1, from the sequence of neighbourhoods of a we can take one $\overline{U}(a, \rho)$ which is contained in O_1 and $\rho<\rho'$. Then since F is closed, we can select from these a finite number of neighbourhoods

$$\overline{U}(a^{(1)}, \rho^{(1)}), \quad \overline{U}(a^{(2)}, \rho^{(2)}), \quad \ldots\ldots, \quad \overline{U}(a^{(q)}), \rho^{(q)}) \tag{5}$$

so that if we denote by T the sum of (5), then $F_1 \leqq T$.[1] Since all the neighbourhoods of (5) are contained in O_1, it is obvious that $T<O_1$.

From the neighbourhoods of (5), first choose one having the greatest ρ and denote it by $\overline{U}(a_1, \rho_1)$ and cancel from (5) those which have any point common with $\overline{U}(a_1, \rho_1)$. Next choose from the remaining neighbourhoods that one which has the greatest ρ, and denote it by $\overline{U}(a_2, \rho_2)$, and so on. After a finite number of such steps we shall have a finite number of neighbourhoods

$$\overline{U}(a_1, \rho_1), \quad \overline{U}(a_2, \rho_2), \quad \ldots\ldots, \quad \overline{U}(a_{p_1}, \rho_{p_1}) \tag{6}$$

Since $\overline{U}(a_1, \rho_1)$ is the one which has the greatest ρ in (5), all the neighbourhoods which are canceled by the first step are contained in $\overline{U}(a_1, 3\rho_1)$ and similarly by the following steps. Therefore

$$T \leqq \overline{U}(a_1, 3\rho_1) \dot{+} \overline{U}(a_2, 3\rho_2) \dot{+} \ldots\ldots \dot{+} \overline{U}(a_{p_1}, 3\rho_{p_1}) .$$

Hence

$$\varphi(T) \leqq \varphi\{\overline{U}(a_1, 3\rho_1)\} + \varphi\{\overline{U}(a_2, 3\rho_2)\} + \ldots\ldots + \varphi\{\overline{U}(a_{p_1}, 3\rho_{p_1})\} .$$

(1) H. Hahn, *loc. cit.*, 89.

Then by (2)

$$\kappa(A, 3)\varphi(T) \leqq \varphi\{\overline{U}(a_1, \rho_1)\} + \varphi\{\overline{U}(a_2, \rho_2)\} + \ldots\ldots + \varphi\{\overline{U}(a_{p_1}, \rho_{p_1})\} ;$$

that is, if we denote by S_1 the sum of (6), we have

$$\kappa(A, 3)\varphi(T) \leqq \varphi(S_1) .$$

Therefore, since $F_1 \subseteqq T$,

$$\kappa(A, 3)\varphi(F_1) \leqq \varphi(S_1) . \tag{7}$$

From (4) and (7) we have

$$\varphi(O_1 - S_1) = \varphi(O_1) - \varphi(S_1) < \left\{1 + \frac{\kappa(A, 3)}{2} - \kappa(A, 3)\right\}\varphi(F_1) .$$

Therefore, if we denote $1 - \frac{\kappa(A, 3)}{2}$ by η, then $\eta < 1$ and

$$\varphi(O_1 - S_1) < \eta\varphi(F_1) \leqq \eta\varphi(A) . \tag{8}$$

Since $\quad A \subseteqq O_1$,

we have $\quad A - AS_1 \subseteqq O_1 - S_1$.

Therefore, by (8)

$$\varphi(A - AS_1) < \eta\varphi(A) . \tag{9}$$

S_1 being closed, $O_1 - S_1$ is open ; and since $A - AS_1$ is normal (φ), there exist as in (4) an open set O_2 and a closed set F_2 so that

$$F_2 \subseteqq A - AS_1 \subseteqq O_2 \subseteqq O_1 - S_1$$

and $$\varphi(O_2) < \left(1 + \frac{\kappa(A, 3)}{2}\right)\varphi(F_2) .$$

Using $A - AS_1$, O_2 and F_2 instead of A_1, O_1, and F_1 in the above process, we shall find a following sequence of neighbourhoods which corresponds to (6)

$$\overline{U}(a_{p_1+1}, \rho_{p_1+1}) , \quad \overline{U}(a_{p_1+2}, \rho_{p_1+2}) ,$$
$$\ldots\ldots , \quad \overline{U}(a_{p_1+p_2}, \rho_{p_1+p_2}) , \tag{10}$$

which are contained in O_2, and if we denote the sum of (10) by S_2, we have as (8)

$$\varphi(O_2-S_2) < \eta\varphi(A-AS_1)\ . \tag{11}$$

Since $S_1 S_2=0$, and

$$A-A(S_1+S_2)=(A-AS_1)-S_2(A-AS_1) \leqq O_2-S_2\,,$$

we have by (9) and (11)

$$\varphi\left\{A-A(S_1+S_2)\right\} < \eta^2\varphi(A)\,.$$

Repeating this process, we have[1]

$$S_3\,,\ S_4\,,\ \ldots\ldots,\ S_n\,,\ \ldots\ldots$$

where

$$S_3=\overline{U}(a_{p_1+p_2+1}\,,\ \rho_{p_1+p_2+1})+\overline{U}(a_{p_1+p_2+2}\,,\ \rho_{p_1+p_2+2})+\ldots\ldots$$
$$+\overline{U}(a_{p_1+p_2+p_3}\,,\ \rho_{p_1+p_2+p_3}),$$

etc.,

and $\qquad S_3 < O_3 < O_2-S_2\,,$

etc.

Let $\qquad S=S_1+S_2+\ldots\ldots+S_n+\ldots\ldots\,,$

then $\qquad S < O_1\,,$

and $\qquad \varphi(A-AS) \leqq \varphi\left\{A-A(S_1+S_2+\ldots\ldots+S_n)\right\} < \eta^n\varphi(A).$

Since $\lim\limits_{n\to\infty}\eta^n=0$, and above inequality holds for any values of n, we have

$$\varphi(A-AS)=0\,. \qquad \text{Q.E.D.}$$

Next by (3), being $\epsilon' < \epsilon$,

$$\varphi(O_1)-\varphi(A) < \epsilon\,.$$

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(1) In some cases $A-A(S_1+S_2+\ldots\ldots+S_n)$ will be empty, then this process ceases at this stage. Let $S=S_1+S_2+\ldots\ldots+S_n$, then since $A=S$, it is obvious that $\varphi(A-AS)=0$, $\varphi(S-AS)=0$. Q.E.D.

Moreover since $\varphi(S) \leqq \varphi(O_1)$ and $\varphi(A) = \varphi(AS)$,

we have $\varphi(S) - \varphi(AS) < \epsilon$,

that is $\varphi(S - AS) < \epsilon$. Q.E.D.

5. Let $\Phi(E)$ and $\varphi(E)$ be two completely additive functions of normal sets in the metric space R, and let $\mathfrak{F}$ be a closed family of sets normal with respect to both $\Phi(E)$ and $\varphi(E)$, which contains all the Borel sets in R. $\Phi(E)$ may have any sign, but assume first that $\varphi(E)$ is *non-negative*.

Now we will define the symmetric derivative as follows: Let a be a point which is not the zero point of $\varphi(E)$. If there exists a sequence

$$\rho_1, \rho_2, \ldots\ldots\ldots, \rho_n, \ldots\ldots\ldots$$

converging to zero, so that

$$\lim_{n \to \infty} \frac{\Phi\{\overline{U}(a, \rho_n)\}}{\varphi\{\overline{U}(a, \rho_n)\}}$$

exists, then this limit is said to be a *symmetric derivative* of $\Phi(E)$, at a, with respect to $\varphi(E)$, and denoted by $D_\varphi^{(s)}\Phi$.

As in §2, we can prove that $\varphi\left\{\overline{U}(a, \rho_n)\right\}$ and $\Phi\left\{\overline{U}(a, \rho_n)\right\}$, considered as functions of the point a, are continuous functions. Therefore, the symetric derivatives $D_\varphi^{(s)}\Phi$ are Baire's functions. Hence when A is a set which belongs to $\mathfrak{F}$, *$D_\varphi^{(s)}\Phi$ are measurable on A with respect to the family $\mathfrak{F}$*, that is the set of points a of A at which $D_\varphi^{(s)}\Phi > \alpha$, belongs to the family $\mathfrak{F}$ for any values of α.

Now assume that $\Phi(E)$ is *absolutely continuous* with respect to $\varphi(E)$,[1] and let A be a set which belongs to the family $\mathfrak{F}$. Denote by A_n the set of points a of A at which $\frac{1}{2^{n-1}} \geqq l(a, \lambda) > \frac{1}{2^n}$. Then

$$A = A_p + A_z + A_1 + A_2 + \ldots\ldots + A_n + \ldots\ldots .$$

Since $l(a, \lambda)$ is a Baire's function, all the sets A_n belong to the family $\mathfrak{F}$, and A_p, A_z are the same as stated in §2.

Since at A_n $\varphi(E)$ is uniformly monotone, we can apply Vitali's theorem. Hence as in the Carathéodory's treatise,[2] for any symmetric derivative $D_\varphi^{(s)}\Phi$ we have

(1) that is, $\Phi(A) = 0$ for any set A when $\varphi(A) = 0$.

(2) Carathéodory, *Vorlesungen über reelle Funktionen*, zweite Aufl. (1927), 484-490.

$$\Phi(A_n)=\int_{A_n} D_\varphi^{(s)}\Phi d\varphi\,.^{(1)}$$

By the additivity of $\Phi(E)$,

$$\Phi(A)=\Phi(A_p)+\Phi(A_z)+\int_{A-A_p-A_z} D_\varphi^{(s)}\Phi d\varphi\,.$$

But as we have seen in §2 $\varphi(A_z)=0$, and hence $\Phi(A_z)=0$, we have

$$\Phi(A)=\Phi(A_p)+\int_{A-A_p} D_\varphi^{(s)}\Phi d\varphi\,.$$

Accordingly $\Phi(E)$ has a unique finite symmetric derivative $D_\varphi^{(s)}\Phi$ almost everywhere (φ) in $A-A_p$.

If a sequence of the sets belonging to the family $\mathfrak{F}$

$$M_1,\ M_2,\ \ldots\ldots,\ M_n,\ \ldots\ldots \tag{1}$$

satisfies the following inequalities

$$\frac{\varphi(M_i)}{\varphi\{\overline{U}(a,\ \rho_i)\}}\geqq\vartheta(a)>0 \qquad (i=1,2,\ \ldots\ldots,\ n,\ \ldots\ldots),$$

where $\overline{U}(a,\ \rho_i)$ is a closed neighbourhood of the point a which contains M_i for $\lim\limits_{i\to\infty}\rho_i=0$, then (1) is called the *φ-regular sequence* of point sets which converges to the point a.

If the φ-regular sequence of point sets (1) be such that

$$\lim_{n\to\infty}\frac{\Phi(M_n)}{\varphi(M_n)}$$

has a limit, then this limit is said to be the *general derivative* of $\Phi(E)$ with respect to $\varphi(E)$ at the point a, and denoted by $D_\varphi\Phi$. Therefore, the values of the general derivatives depend on the φ-regular sequence of point sets. But as in the Carathéodory's treatise,[2] we can prove that *when $\Phi(E)$ is absolutely continuous with respect to $\varphi(E)$, $\Phi(E)$ has a unique finite general derivative $D_\varphi\Phi$ almost everywhere (φ) in $A-A_p$, and*

(1) For the integral of a point function with respect to a set function in a metric space, cf. Fréchet, *Bull. de la Soc. Math. de France*, **43** (1915), 248-265.

(2) *Loc. cit.*, 493-496.

$$\Phi(A)=\Phi(A_p)+\int_{A-A_p} D_\varphi \Phi d\varphi\,,$$

for any set A of $\mathfrak{F}$.

6. To investigate the case where $\Phi(E)$ is *not absolutely continuous* with respect to $\varphi(E)$, we have first to prove the following theorem:

When $\Phi(E)$ has any sign and vanishes for all parts B of A, which belong to $\mathfrak{F}$, then all the general derivatives $D_\varphi \Phi$ are zero almost everywhere (φ) in $A-A_p$.

Consider first the positive variation $\Phi^+(E)$ of $\Phi(E)$, then $\Phi^+(B)=0$ for all parts B of A. Let

$$M_1,\ M_2,\ \ldots\ldots,\ M_n,\ \ldots\ldots$$

be a φ-regular sequence of sets, so that

$$\frac{\varphi(M_n)}{\varphi\{\overline{U}(a,\rho_n)\}}\geqq\vartheta(a)>0\,.$$

Then $$\frac{\Phi^+(M_n)}{\varphi(M_n)}\leqq\frac{\varphi\{\overline{U}(a,\rho_n)\}}{\varphi(M_n)}\,\frac{\Phi^+\{\overline{U}(a,\rho_n)\}}{\varphi\{\overline{U}(a,\rho_n)\}}\leqq\frac{1}{\vartheta(a)}\,\frac{\Phi^+\{\overline{U}(a,\rho_n)\}}{\varphi\{\overline{U}(a,\rho_n)\}}\,.$$

Therefore, if all the symmetric derivatives $D_\varphi^{(s)}\Phi^+$ vanish at a, then the general derivatives $D_\varphi\Phi^+$ vanish also. Therefore, it is sufficient to prove this theorem for the symmetric derivatives $D_\varphi^{(s)}\Phi^+$.

Assume that the symmetric derivatives $D_\varphi^{(s)}\Phi^+$ are not zero almost everywhere (φ) in $A-A_p$, then there exists a part B of A, so that $\varphi(E)$ is uniformly monotone at B, $\varphi(B)>0$, and $D_\varphi^{(s)}\Phi^+>\alpha$ at any point of B, where α is a positive number sufficiently small. Then at any point b of B, there exists a sequence of neighbourhoods

$$\overline{U}(b,\rho^{(1)}),\ \overline{U}(b,\rho^{(2)}),\ \ldots\ldots,\ \overline{U}(b,\rho^{(n)}),\ \ldots\ldots$$

so that

$$\Phi^+\left\{\overline{U}(b,\rho^{(i)})\right\}>(\alpha-\epsilon)\varphi\left\{\overline{U}(b,\rho^{(i)})\right\}\quad(i=1,2,\ldots\ldots,n,\ldots\ldots),\quad(1)$$

where ϵ is a given positive number less than α. Let O be any open set containing B; then by Vitali's theorem there exists a sum of neighbourhoods

$$S = \overline{U}(b_1, \rho_1) + \overline{U}(b_2, \rho_2) + \ldots\ldots + \overline{U}(b_n, \rho_n) + \ldots\ldots$$

such that $\qquad S < 0$,

and $\qquad \varphi(B - BS) = 0$.

But by (1) $\qquad \Phi^+(S) > (\alpha - \epsilon)\varphi(S)$. $\qquad$ (2)

Hence by (2) $\qquad \Phi^+(S) > (\alpha - \epsilon)\varphi(B)$. $\qquad$ (3)

But since $\Phi^+(B) = 0$, and the Φ-normal set is also Φ^+-normal, we may take O so that $\Phi^+(O)$ is less than any positive number. On the other hand since $\Phi^+(S) \leqq \Phi^+(O)$, by the inequality (3) we conclude that $\varphi(B) = 0$. This is absurd. Therefore, $D_\varphi^{(s)}\Phi^+ = 0$ almost everywhere (φ) in $A - A_p$. Hence by the above we may conclude that $D_\varphi \Phi^+ = 0$ almost everywhere (φ) in $A - A_p$; the same thing holds true for $D_\varphi \Phi^-$. Therefore, $D_\varphi \Phi = 0$ almost everywhere (φ) in $A - A_p$.

Now as the set function $\Phi(E)$ is not absolutely continuous with respect to $\varphi(E)$, we can decompose any set A of $\mathfrak{F}$ into its regular-part $A^\times$ and singular-part $A^{\times\times}$, both belonging to $\mathfrak{F}$, so that

$$A = A^\times + A^{\times\times},$$

and that denoting

$$\Phi^\times(A) = \Phi(A^\times) \quad \text{and} \quad \Phi^{\times\times}(A) = \Phi(A^{\times\times}),$$

we have $\qquad \Phi(A) = \Phi^\times(A) + \Phi^{\times\times}(A)$,

where $\Phi^\times(A)$ and $\Phi^{\times\times}(A)$ are the functions of regularity and of singularity of $\Phi(A)$ respectively.(1)

Now $\Phi^{\times\times}(B) = 0$ for all parts B of $A^\times$, therefore, by the above theorem the general derivatives $D_\varphi \Phi^{\times\times}$ are zero almost everywhere (φ) in $A^\times - A_p^\times$.(2) Hence they are zero almost everywhere (φ) in $A - A_p^\times$, since $\varphi(A^{\times\times}) = 0$.(3)

Since the function $\Phi^\times(E)$ is absolutely continuous with respect to $\varphi(E)$, by § 5 $\Phi^\times(E)$ has a unique finite general derivative $D_\varphi \Phi^\times$ with respect to $\varphi(E)$ almost everywhere (φ) in $A - A_p$, that is, in $A - A_p^\times$ and

(1) H. Hahn, *loc. cit.*, 419-421.

(2) $A_p^\times$ means the set of poles in $A^\times$.

(3) This can be proved directly from the definition of $A^{\times\times}$.

$$\Phi^{\times}(A)=\Phi^{\times}(A_p)+\int_{A-A_p^{\times}} D_{\varphi}\Phi^{\times}d\varphi\,.$$

But since $\qquad \Phi(A)=\Phi^{\times}(A)+\Phi^{\times\times}(A),$

combining it with the above result, we have the following theorem:

When $\Phi(E)$ is not absolutely continuous with respect to $\varphi(E)$, $\Phi(E)$ has a unique finite general derivative $D_{\varphi}\Phi$ almost everywhere (φ) in $A-A_p^{\times}$, and

$$\Phi(A)=\Phi(A^{\times\times})+\Phi(A_p^{\times})+\int_{A-A_p^{\times}} D_{\varphi}\Phi d\varphi\,,$$

for any set A of $\mathfrak{F}$.

7. When the fnnction $\varphi(E)$ has *any sign*, the definition of the general derivatives $D_{\varphi}\Phi$ of $\Phi(E)$ with respect to $\varphi(E)$ is the same as in § 5. But for the φ-regular sequence we use the condition

$$\frac{\bar{\varphi}(M_i)}{\bar{\varphi}\{\bar{U}(a,\rho_i)\}}\geqq\vartheta(a)>0\,,\ \text{and}\ \varphi(M_i)\neq 0,\ (i=1,2,\ldots\ldots,n,\ldots\ldots).$$

When A is a set of the family $\mathfrak{F}$, we decompose A into its positive part A^+ and negative part A^-, and consider a point function $h(a)$ so that

$$\begin{aligned} h(a)&=1 && \text{when } a \text{ is a point of } A^+,\\ &=-1 && \quad ,, \quad ,, \quad ,, \quad A^-, \end{aligned}$$

then $h(a)$ is summable $(\bar{\varphi})$ and

$$\int_A h(a)d\bar{\varphi}=\int_{A^+} d\bar{\varphi}-\int_{A^-} d\bar{\varphi}=\bar{\varphi}(A^+)-\bar{\varphi}(A^-).$$

Therefore, we have

$$\int_A h(a)d\bar{\varphi}=\varphi(A).$$

Then from § 5 $D_{\bar{\varphi}}\varphi$ is equivalent (φ)[1] to $h(a)$ in $A-A_p$.

Let $M_1, M_2, \ldots\ldots, M_n, \ldots\ldots$ be a φ-regular sequence. Since

(1) When $\varphi(E)$ has any sign, equivalent (φ) means equivalent $(\bar{\varphi})$.

$\varphi(M_n) \neq 0$ and $\bar{\varphi}(M_n) \neq 0$, we have

$$\frac{\Phi(M_n)}{\varphi(M_n)} = \frac{\dfrac{\Phi(M_n)}{\bar{\varphi}(M_n)}}{\dfrac{\varphi(M_n)}{\bar{\varphi}(M_n)}}.$$

As n becomes infinite, $\dfrac{\Phi(M_n)}{\bar{\varphi}(M_n)}$ has a finite limit $D_{\bar{\varphi}}\Phi$ almost everywhere (φ) in $A-A_p^{\times}$, and $\dfrac{\varphi(M_n)}{\bar{\varphi}(M_n)}$ has a limit $h(a)$ almost everywhere (φ) in $A-A_p$, that is, in $A-A_p^{\times}$. Therefore, $\dfrac{\Phi(M_n)}{\varphi(M_n)}$ has a limit $\dfrac{D_{\bar{\varphi}}\Phi}{h(a)}$ almost everywhere (φ) in $A-A_p^{\times}$, that is, $D_{\varphi}\Phi$ is equivalent (φ) to $D_{\bar{\varphi}}\Phi$ in $A^+-A^+A_p^{\times}$ and to $-D_{\bar{\varphi}}\Phi$ in $A^--A^-A_p^{\times}$. Adding to it, we have

$$\begin{aligned}\Phi(A)-\Phi(A^{\times\times})-\Phi(A_p^{\times}) &= \int_{A-A_p^{\times}} D_{\bar{\varphi}}\Phi d\bar{\varphi} \\ &= \int_{A^+-A^+A_{\varphi}^{\times}} D_{\varphi}\Phi d\varphi + \int_{A^--A^-A_p^{\times}} D_{\varphi}\Phi d\varphi \\ &= \int_{A-A_p^{\times}} D_{\varphi}\Phi d\varphi .\end{aligned}$$

Now we have the following conclusion:

When $\Phi(E)$ and $\varphi(E)$ have any sign, $\Phi(E)$ has a unique finite general derivative $D_{\varphi}\Phi$ almost everywhere (φ) in $A-A_p^{\times}$, which is equivalent (φ) to $D_{\bar{\varphi}}\Phi$ in $A^+-A^+A_p^{\times}$, and to $-D_{\bar{\varphi}}\Phi$ in $A^--A^-A_p^{\times}$, and

$$\Phi(A)=\Phi(A^{\times\times})+\Phi(A_p^{\times})+\int_{A-A_p^{\times}} D_{\varphi}\Phi d\varphi ,$$

for any set A of $\mathfrak{F}$.

Moreover $\displaystyle\int_{A-A_p^{\times}} D_{\varphi}\Phi d\varphi = \int_{A-A_{\varphi}^{\times}} h(a) D_{\varphi}\Phi d\bar{\varphi}$ [1]

(1) This corresponds to the property of the Stieltjes integral with respect to a point function. Cf. Lebesgue, *Leçons sur l'intégration*, (1928), 260.

where $h(a)$ *is equivalent* (φ) *to* 1 *in* $A^+ - A^+A_p^\times$ *and to* -1 *in* $A^- - A^-A_p^\times$.

General Derivatives in the Euclidian Space

8. In the Euclidian space R_n of n dimensions, we can use de la Vallée Poussin's conjugate systems of nets, and we obtain the above results of § 6 where $\varphi(E)$ is *non-negative*, more easily.

Let

$$W_1' \supset W_2' \supset \ldots\ldots \supset W_n' \supset \ldots\ldots$$

be a sequence of meshes of a system of nets which contain the point a, then the upper limit and the lower limit of $\dfrac{\Phi(W_n')}{\varphi(W_n')}$, as n becomes infinite, are called respectively the *upper* and *lower net derivative* of $\Phi(E)$ with respect to $\varphi(E)$ at a on the given systems of nets, and we denote these by the symbols $\overline{D}_\varphi^{(n)}\Phi$ and $\underline{D}_\varphi^{(n)}\Phi$; when $\overline{D}_\varphi^{(n)}\Phi = \underline{D}_\varphi^{(n)}\Phi$, we denote this by $D_\varphi^{(n)}\Phi$. It is obvious that $\overline{D}_\varphi^{(n)}\Phi$ and $\underline{D}_\varphi^{(n)}\Phi$ are measurable (B).[1]

The symmetric derivative $D_\varphi^{(s)}\Phi$ and the general derivative $D_\varphi\Phi$ may be also defined similarly as in § 5, using the closed intervals $\overline{W}(a, \rho)$ instead of the closed neighbourhoods $\overline{U}(a, \rho)$.

At the zero points of A with respect to $\varphi(E)$, the above definitions of net derivatives are meaningless. But in the following, it is convenient to assign to them certain values. For this purpose, we decompose the set A_z into two sets $A_z^\times$ and $A_z^{\times\times}$, where $A_z^\times$ is the set of the points which are also the zero points of $\Phi(E)$, and $A_z^{\times\times}$ is the set of the points which are not the zero points of $\Phi(E)$. Then assign to the net derivatives the value 0 for the points of $A_z^\times$, and infinity for the points of $A_z^{\times\times}$, the sign of infinity being positive at the positive part of $A_z^{\times\times}$, and negative at the negative part of $A_z^{\times\times}$ with respect to $\Phi(E)$. By § 2, $\varphi(A_z)=0$ and $\Phi(A_z^\times)=0$.

9. We shall prove the fundamental lemma:

If at any point of a set A *of* $\mathfrak{F}$, *the net derivative, for example* $\overline{D}_\varphi^{(n)}\Phi$ *is such that*

$$\alpha \leqq \overline{D}_\varphi^{(n)}\Phi \leqq \beta\,,$$

(1) De la Vallée Poussin, *loc. cit.*, 63.

then $$\alpha\varphi(A) \leqq \Phi(A) \leqq \beta\varphi(A) .$$

Since A is normal with respect to $\Phi(E)$ and $\varphi(E)$, we can find an open set O and a closed set F so that $F \subseteqq A \subseteqq O$ and for any given positive number ϵ

$$\varphi(O)-\varphi(F) < \epsilon ,$$

and $$|\Phi(A)-\Phi(S)| < \epsilon ,$$

where S is any set of $\mathfrak{F}$ satisfying the condition $F \subseteqq S \subseteqq O$.

For any point a of A there exists a sequence of meshes

$$W_1'(a) \supset W_2'(a) \supset \ldots\ldots \supset W_i'(a) \supset \ldots\ldots$$

which contain the point a, and

$$\alpha-\epsilon < \frac{\Phi\{W_i'(a)\}}{\varphi\{W_i'(a)\}} < \beta+\epsilon . \tag{1}$$

From these we can find an enumerably infinite number of meshes

$$W'(a_1), \quad W'(a_2), \ \ldots\ldots , \quad W'(a_n), \ \ldots\ldots \tag{2}$$

which have no common points, and if we denote the sum of (2) by S, i.e.

$$S = W'(a_1) + W'(a_2) + \ldots\ldots + W'(a_n) + \ldots\ldots ,$$

then $$F \subseteqq S \subseteqq O .$$

Hence by (1) $$(\alpha-\epsilon)\varphi(S) < \Phi(S) < (\beta+\epsilon)\varphi(S) ,$$

where $$|\Phi(A)-\Phi(S)| < \epsilon ,$$

and $$|\varphi(A)-\varphi(S)| < \epsilon .$$

Let ϵ converge to zero, then we have

$$\alpha\varphi(A) \leqq \Phi(A) \leqq \beta\varphi(A) .$$

The above proof also holds good for the net derivatives at the zero points of $\varphi(E)$ by a slight modification.

From this, as Schlesinger and Plessner did,[1] we have the following theorem:

$\Phi(E)$ has a unique finite net derivative $D_\varphi^{(n)}\Phi$ almost everywhere (φ) in A, and

$$\Phi(A)=\Phi(A^{\times\times})+\int_A D_\varphi^{(n)}\Phi d\varphi\ .$$

$A^{\times\times}$ is the so-called singular-part of A and $A^{\times}=A-A^{\times\times}$ is the regular-part of A for $\Phi(E)$ with respect to $\varphi(E)$.

10. To investigate the properties of the general derivatives we must first prove the following lemma:

Let $\Phi(E)$ be non-negative, and if all the net derivatives $D_\varphi^{(n)}\Phi$ on the conjugate systems[2] *are zero at a point a which is not a pole of $\varphi(E)$, then all the general derivatives $D_\varphi\Phi$ are zero at that point.*

To prove this, let M be a set which belongs to the φ-regular sequence of point sets at a, so that

$$\frac{\varphi(M)}{\varphi\{\overline{W}(a,\rho)\}}\geqq\vartheta(a)>0\ ,$$

then there exists a mesh W' with its sides equal to σ and belonging to one of the conjugate systems, so that $2\rho<\sigma\leqq 6\rho$, and

$$\overline{W}(a,\rho)\subset W'\subset\overline{W}\left(a,\frac{3}{2}\sigma\right).$$

But since $\quad \overline{W}\left(a,\frac{3}{2}\sigma\right)\subseteqq\overline{W}(a,9\rho)\ ,$

we have $\quad \overline{W}(a,\rho)\subset W'\subset\overline{W}(a,9\rho)\ .$

Therefore, we have

$$\frac{\Phi(M)}{\varphi(M)}\leqq\frac{1}{\vartheta(a)}\frac{\Phi\{\overline{W}(a,\rho)\}}{\varphi\{\overline{W}(a,\rho)\}}\leqq\frac{1}{\vartheta(a)}\frac{\varphi\{\overline{W}(a,9\rho)\}}{\varphi\{\overline{W}(a,\rho)\}}\frac{\Phi(W')}{\varphi(W')}\ .$$

since $\quad \varphi\left\{\overline{W}(a,\rho)\right\}\geqq\kappa(a,9)\varphi\left\{\overline{W}(a,9\rho)\right\}\ ,$

(1) *Lebesgueshe Integrale und Fouriersche Reihen,* (1926), 158-160.

(2) De la Vallée Poussin, *loc. cit.*, 64.

we have $$\frac{\Phi(M)}{\varphi(M)} \leqq \frac{1}{\vartheta(a)\kappa(a, 9)} \frac{\Phi(W')}{\varphi(W')} .$$

Since $\lim_{\sigma\to 0} \frac{\Phi(W')}{\varphi(W')}$ is a net derivative at a and zero by hypothesis, we conclude that the general derivative $\lim_{\rho\to 0} \frac{\Phi(M)}{\varphi(M)}$ at a is zero.

Consequently we have the following theorem:

When $\Phi(E)$ has any sign and vanishes for all parts B of A which belong to $\mathfrak{F}$, then the general derivatives $D_\varphi \Phi$ are zero almost everywhere (φ) in $A-A_p$.

In this case the positive variation $\Phi^+(E)$ of $\Phi(E)$ vanishes for all parts B of A. Hence all the net derivatives of $\Phi^+(E)$ on the conjugate systems are zero almost everywhere (φ) in A. For if otherwise, there would exist a part B at which a net derivative $D_\varphi^{(n)}\Phi^+ > \alpha$ where α is a positive number sufficiently small, and $\varphi(B) > 0$. Then by § 9,

$$\Phi^+(B) \geqq \alpha\varphi(B) > 0 .$$

This is absurd. Hence by the above lemma, $D_\varphi\Phi^+$ are zero almost everywhere (φ) in $A-A_p$.

Similarly for $D_\varphi\Phi^-$, therefore, the general derivatives $D_\varphi \Phi$ are zero almost everywhere (φ) in $A-A_p$.

11. From the theorem of § 9, the functions of singularity $\Phi^{\times\times}(E)$ and of regularity $\Phi^\times(E)$ of $\Phi(E)$ are represented as follows:

$$\Phi^{\times\times}(A) = \Phi(A^{\times\times}) ,$$

$$\Phi^\times(A) = \int_A D_\varphi^{(n)}\Phi d\varphi .$$

First consider the function of singularity $\Phi^{\times\times}(E)$. Since $\Phi^{\times\times}(B)=0$ for all parts B of $A^\times$, by § 10 the general derivatives $D_\varphi \Phi^{\times\times}$ vanish almost everywhere (φ) in $A^\times - A_p^\times$.

Next for the function of regularity

$$\Phi^\times(A) = \int_A D_\varphi^{(n)}\Phi d\varphi ,$$

by the method of de la Vallée Possin,[1] we can prove that all the general derivatives $D_\varphi \Phi^\times$ are equivalent (φ) to $D_\varphi^{(n)}\Phi$ in $A-A_p$.

(1) *Loc. cit.*, 71-72.

Therefore, we have

$$\Phi^{\times}(A)=\Phi^{\times}(A_p)+\int_{A-A_p} D_{\varphi}\Phi^{\times}d\varphi\ .$$

Since $D_{\varphi}\Phi=D_{\varphi}\Phi^{\times\times}+D_{\varphi}\Phi^{\times}$ and $\varphi(A^{\times\times})=0$, $\Phi(E)$ has a unique general derivative $D_{\varphi}\Phi$ almost everywhere (φ) in $A-A_p^{\times}$. And since $D_{\varphi}\Phi$ is equivalent (φ) to $D_{\varphi}\Phi^{\times}$ in $A-A_p^{\times}$ and $\Phi^{\times}(A_p)=\Phi(A_p^{\times})$, we have

$$\Phi(A)=\Phi^{\times\times}(A)+\Phi^{\times}(A)=\Phi(A^{\times\times})+\Phi(A_p^{\times})+\int_{A-A_p^{\times}} D_{\varphi}\Phi d\varphi\ .$$

Therefore, we have the theorem of § 6.

Thus *in the Euclidian space we have the same conclusion as in the metric space using different methods of proof.*

12. In the Euclidian space we can investigate the properties of the general derivatives at the points of $A_p^{\times}$ more in detail. For this purpose suppose first that $\Phi(E)$ and $\varphi(E)$ are *both non-negative.*

If at a point a of $A_p^{\times}$ there exists a unique symmetric derivative $D_{\varphi}^{(s)}\Phi$ neither zero nor infinity, then

$$\lim_{\rho\to 0}\frac{\dfrac{\Phi\{\overline{W}(a,\ \rho)\}}{\varphi\{\overline{W}(a,\ \rho)\}}}{\dfrac{\Phi\{\overline{W}(a,\ \lambda\rho)\}}{\varphi\{\overline{W}(a,\ \lambda\rho)\}}}=1\ ,$$

that is
$$\lim_{\rho\to 0}\frac{\dfrac{\Phi\{\overline{W}(a,\ \rho)\}}{\Phi\{\overline{W}(a,\ \lambda\rho)\}}}{\dfrac{\varphi\{\overline{W}(a,\ \rho)\}}{\varphi\{\overline{W}(a,\ \lambda\rho)\}}}=1\ .$$

Therefore, we must have

$$\lim_{\rho\to 0}\frac{\Phi\{\overline{W}(a,\ \rho)\}}{\Phi\{\overline{W}(a,\ \lambda\rho)\}}=\lim_{\rho\to 0}\frac{\varphi\{\overline{W}(a,\ \rho)\}}{\varphi\{\overline{W}(a,\ \lambda\rho)\}}\ .$$

But since a is the pole of $\varphi(E)$, we have

$$\lim_{\rho\to 0}\frac{\varphi\{\overline{W}(a,\ \rho)\}}{\varphi\{\overline{W}(a,\ \lambda\rho)\}}=0\ ;$$

therefore,
$$\lim_{\rho\to 0}\frac{\Phi\{\overline{W}(a,\ \rho)\}}{\Phi\{\overline{W}(a,\ \lambda\rho)\}}=0\ ;$$

that is, a is the pole of $\Phi(E)$. Therefore, except in the case of zero or infinity, $\Phi(E)$ can not have a unique symmetric derivative in $A_p^\times - A_p^\times A_{(p)}^\times$, where $A_{(p)}^\times$ means the set of poles of $\Phi(E)$ in $A^\times$.

Denote the regular and singular-parts of A for $\varphi(E)$ with respect to $\Phi(E)$ by $A^{(\times)}$ and $A^{(\times\times)}$ respectively, then $\Phi(A^{(\times\times)})=0$. In the conclusion of § 11, if we interchange $\varphi(E)$ with $\Phi(E)$, then $\varphi(E)$ has a unique finite symmetric derivative $D_\Phi^{(s)}\varphi$ in $A^{(\times)}-A_{(p)}^{(\times)}$, except the set H for which $\Phi(H)=0$. Since H is contained in the regular-part $A^{(\times)}$ of A, $\varphi(H)=0$; therefore, $\varphi(E)$ has a unique finite symmetric derivative $D_\Phi^{(s)}\varphi$ almost everywhere (φ) in $A^{(\times)}-A_{(p)}^{(\times)}$. But since the set of points at which $D_\Phi^{(s)}\varphi$ is zero, has the φ-value zero, $\varphi(E)$ has a unique finite, non-zero, symmetric derivative $D_\Phi^{(s)}\varphi$ almost everywhere (φ) in $A^{(\times)}-A_{(p)}^{(\times)}$. Since $D_\varphi^{(s)}\Phi=\dfrac{1}{D_\Phi^{(s)}\varphi}$, $\Phi(E)$ has a unique finite, non-zero, symmetric derivative $D_\varphi^{(s)}\Phi$ almost everywhere (φ) in $A^{(\times)}-A_{(p)}^{(\times)}$. But as we have seen, in

$$\begin{aligned} A_p^\times - A_p^\times A_{(p)}^\times &= (A_p^\times - A_p^\times A_{(p)}^\times)A^{(\times\times)} + (A_p^\times - A_p^\times A_{(p)}^\times)A^{(\times)} \\ &= (A_p^\times - A_p^\times A_{(p)}^\times)A^{(\times\times)} + A_p^\times(A^{(\times)} - A_p^{(\times)}) \end{aligned}$$

$\Phi(E)$ can not have a unique symmetric derivative $D_\varphi^{(s)}\Phi$ except in the case of zero or infinity. Therefore, it must be $\varphi\left\{A_p^\times(A^{(\times)}-A_{(p)}^{(\times)})\right\}=0$. But since $\varphi(A_p^{\times\times})=0$, we have

$$\varphi\left\{A_p(A^{(\times)}-A_{(p)}^{(\times)})\right\}=0\ . \tag{1}$$

The above consideration also holds for the Lebesgue measure $m(E)$ instead of $\Phi(E)$. It is obvious that $m(E)$ has no poles. If we denote the regular and singular-parts of A for $\varphi(E)$ with respect to $m(E)$ by $A^{((\times))}$ and $A^{((\times\times))}$ respectively, then by (1) we have

$$\varphi(A_p A^{((\times))})=0\ . \tag{2}$$

Now let $\varphi(E)$ have any sign. When $\varphi(E)$ is absolutely continuous with respect to $m(E)$, then $\bar{\varphi}(E)$ is so also, and the poles of $\varphi(E)$ are, by the definition, the poles of $\bar{\varphi}(E)$. Therefore, by (2) we have the following theorem:

When $\varphi(E)$ has any sign, and is absolutely continuous with respect to $m(E)$, then

$$\bar{\varphi}(A_p)=0$$

for any set A *of* $\mathfrak{F}$.

Consequently the theorem of § 7 may be stated as follows:

When $\Phi(E)$ *and* $\varphi(E)$ *have any sign, and if* $\varphi(E)$ *is absolutely continuous with with respect to* $m(E)$, *then* $\Phi(E)$ *has a unique finite general derivative* $D_\varphi \Phi$ *almost everywhere* (φ) *in* A,[1] *and*

$$\Phi(A)=\Phi(A^{\times\times})+\int_A D_\varphi \Phi d\varphi$$

for any set A *of* $\mathfrak{F}$.

13. Now we prove the following lemma which is similar to that of § 10.

Let $\Phi(E)$ *and* $\varphi(E)$ *be non-negative, and if all the net derivatives* $D_\varphi^{(n)}\Phi$ *on the conjugate systems are zero at a point* a *not a pole of* $\Phi(E)$, *then all the general derivatives* $D_\varphi \Phi$ *are zero at that point.*

Let M be a set belonging to the φ-regular sequence of point sets, so that

$$\frac{\varphi(M)}{\varphi\{\bar{W}(a,\rho)\}}\geqq\vartheta(a)>0\ ,$$

then there exists a mesh W' with its sides equal to σ, and belonging to one of the conjugate systems, where $\frac{2}{9}\rho\leqq\sigma<\frac{2}{3}\rho$, so that

$$\bar{W}\left(a,\ \frac{\sigma}{2}\right)<W'<\bar{W}(a,\rho)\ .$$

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(1) This can be proved directly, since

$$\frac{\Phi(M_n)}{\varphi(M_n)}=\frac{\dfrac{\Phi(M_n)}{m(M_n)}}{\dfrac{\varphi(M_n)}{m(M_n)}},$$

and φ-regular sequence of point sets is m-regular almost everywhere (φ) in A; for, if otherwise, there exists a partial saquence so that $\frac{\bar{\varphi}(M_i)}{\bar{\varphi}(\bar{W}_i)}\geqq\vartheta$ and $\frac{m(M_i)}{m(\bar{W}_i)}<\eta$, where ϑ is constant and $\lim_{i\to\infty}\eta=0$, and then $\dfrac{\dfrac{\bar{\varphi}(M_i)}{m(M_i)}}{\dfrac{\bar{\varphi}(\bar{W}_i)}{m(\bar{W}_i)}}>\frac{\vartheta}{\eta}$. Therefore, at such points there is no unique general derivative $D_m\bar{\varphi}$ except in the case of zero or infinity; that is, the set of such points in A has φ-value zero.

But since $$\overline{W}\left(a, \frac{\rho}{9}\right) \leqq \overline{W}\left(a, \frac{\sigma}{2}\right),$$

we have $$\overline{W}\left(a, \frac{\rho}{9}\right) < W' < (a, \rho) .$$

Therefore, we have

$$\frac{\Phi(M)}{\varphi(M)} \leqq \frac{1}{\vartheta(a)} \frac{\Phi\{\overline{W}(a, \rho)\}}{\varphi\{\overline{W}(a, \rho)\}} \leqq \frac{1}{\vartheta(a)} \frac{\Phi\{\overline{W}(a, \rho)\}}{\Phi\left\{\overline{W}\left(a, \frac{\rho}{9}\right)\right\}} \frac{\Phi(W')}{\varphi(W')} .$$

But $$\Phi\left\{\overline{W}\left(a, \frac{\rho}{9}\right)\right\} \geqq \kappa'(a, 9)\, \Phi\left\{\overline{W}(a, \rho)\right\},$$

therefore $$\frac{\Phi(M)}{\varphi(M)} \leqq \frac{1}{\vartheta(a)\kappa'(a, 9)} \frac{\Phi(\overline{W}')}{\varphi(\overline{W}')} .$$

Since $\lim\limits_{\sigma\to 0} \dfrac{\Phi(W')}{\varphi(W')}$ is the net derivative at a and it is zero, the general derivative $\lim\limits_{\rho\to 0} \dfrac{\Phi(M)}{\varphi(M)}$ at a is zero.

Consequently as in § 10, we have the following theorem :

When $\Phi(E)$ vanishes for all parts B of A, which belongs to $\mathfrak{F}$, then the general derivatives $D_\varphi \Phi$ are zero almost everywhere (φ) in $A - A_{(p)}$.

Now since the non-negative function $\Phi(E)$ vanishes for all parts B of $A_p^\times A^{(\times\times)}$, the general derivatives $D_\varphi \Phi$ are zero almost everywhere (φ) in $(A_p^\times - A_p^\times A_{(p)}^\times)A^{(\times\times)}$. But from (1) in § 12 $\varphi\{A_p^\times(A^{(\times)} - A_{(p)}^{(\times)})\} = 0$, that is, $\varphi\{(A_p^\times - A_p^\times A_{(p)}^\times)A^{(\times)}\} = 0$, we can say that the general derivatives $D_\varphi \Phi$ are zero almost everywhere (φ) in $A_p^\times - A_p^\times A_{(p)}^\times$ and

$$\int_{A_p^\times - A_p^\times A_{(p)}^\times} D_\varphi \Phi d\varphi = 0 .$$

From $\varphi\{(A_p^\times - A_p^\times A_{(p)}^\times)A^{(\times)}\} = 0$, we have $\Phi\{(A_p^\times - A_p^\times A_{(p)}^\times)A^{(\times)}\} = 0$. But $\Phi(A^{(\times\times)}) = 0$; therefore, $\Phi(A_p^\times - A_p^\times A_{(p)}^\times) = 0$; that is, $\Phi(A_p^\times) = \Phi(A_p^\times A_{(p)}^\times)$. From (2) in § 12, we have $\varphi(A_p^\times A^{((\times))}) = 0$ and $\Phi(A_p^\times A^{((\times))}) = 0$.

Combining these results with that of § 11, we have the following conclusion:

When $\Phi(E)$ and $\varphi(E)$ are both non-negative, and if $\varphi(E)$ is not absolutely continuous with respect to $m(E)$, $\Phi(E)$ have a unique finite general derivative $D_\varphi \Phi$ almost everywhere (φ) in A except $A_p^\times A_{(p)}^\times A^{((\times\times))}$, of Lebesgue measure zero, and we have

$$\Phi(A)=\Phi(A^{\times\times})+\Phi(A_p^\times A_{(p)}^\times A^{((\times\times))})+\int_{A-A_p^\times A_{(p)}^\times A^{((\times\times))}} D_\varphi \Phi d\varphi\,,$$

for any set A belonging to $\mathfrak{F}$.