Mathematical Foundations of Quantum Mechanics.

By

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Although quantum mechanics has achieved great success in the explanation of physical phenomena, the mathematics used in quantum mechanics is not rigorous. Unrigorous treatments occur in the case of continuous spectrum. In a Hilbert space, let A be a self-adjoint operator. When we define the eigenvalue and the eigenelement by the relation

$$A\mathfrak{f}=\lambda\mathfrak{f}, \tag{1}$$

we encounter no obstacle in the case of discrete eigenvalues. But if we use the same definition in the case of continuous eigenvalues we encounter a serious obstacle. Let $\mathfrak{f}_\lambda$ be the eigenelement which corresponds to the continuous eigenvalue λ; then the orthogonality of the system of eigenelements is expressed as follows:

$$(\mathfrak{f}_\lambda, \mathfrak{f}_{\lambda'})=\delta(\lambda-\lambda'),$$

where δ is the Dirac improper δ function, defined by

$$\int_{-\infty}^{\infty}\delta(x)\,dx=1$$

$$\delta(x)=0 \qquad \text{for} \quad x\neq 0.$$

Thus we must deal with an improper function. Furthermore, the eigenelements $\mathfrak{f}_\lambda$ do not exist in the Hilbert space.(1)

For example, let the Hilbert space be the space of point functions $f(q)$ (i. e. ordinary wave functions). In the case of the operator which corresponds to the coordinate,

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(1) Cf. P. A. M. Dirac [2], 78–79. The numbers in square brackets refer to the bibliography at the end of the paper,

$$qf(q)=\lambda f(q)$$

has solutions
$$f(q)=\delta(q-\lambda)$$

for any λ, which do not exist. And in the case of the operator which corresponds to the momentum,

$$\frac{h}{2\pi i}\frac{d}{dq}f(q)=\lambda f(q)$$

has solutions
$$f(q)=ce^{\frac{2\pi i}{h}\lambda q}$$

for any λ. But these are not quadratically integrable.

E. Hellinger and H. Weyl(1) have already investigated the problem of continuous spectrum and introduced the "eigendifferential." But they are concerned with a special problem, i. e. bounded quadratic forms and solutions of differential equation. In respect of generality, they do not satisfy.(2) E. Fues(3) and some other writers(4) have used this conception "eigendifferential" in quantum mechanics. But the treatments are either complicated or not rigorous.

J. v. Neumann(5) has proved that in Hilbert space, for any self-adjoint operator $\boldsymbol{A}$, there exists a resolution of identity $\boldsymbol{E}(\lambda)$ such that

$$\boldsymbol{A}\mathfrak{f}=\int_{-\infty}^{\infty}\lambda\boldsymbol{E}(d\lambda)\mathfrak{f}, \tag{2}$$

and he has applied this theory to quantum mechanics. Although his theory gave a rigorous mathematical foundation to quantum mechanics, it did not give any interpretation to the representation theory of Dirac's form.

Recently I investigated the "theory of vector-valued set functions." This theory removes the difficulties in the treatement of continuous spectrum, and gives a rigorous interpretation to the representation theory.(6) The vector-valued set function $\mathfrak{q}(U)$ is a function of set U, and its functional value is an element (vector) in the Hilbert space,

(1) E. Hellinger [1], 231–271. H. Weyl [1], 238–251.

(2) T. Carlemann used this conception "eigendifferential" in the treatise of singular integral equations. Cf. T. Carlemann [1].

(3) E. Fues [1], 295–303.

(4) For example, L. de Broglie [2], 140–159.

(5) J. v. Neumann [1], 53–121.

(6) F. Maeda [2], [5], [6]. A summary of the theory of vector-valued set functions is given in the first part of the present paper.

which satisfies the following conditions:

$$\big(\mathfrak{q}(U), \mathfrak{q}(U')\big)=0 \quad \text{when} \quad UU'=0\,,$$

and

$$\mathfrak{q}(U)=\sum_i \mathfrak{q}(U_i) \quad \text{when} \quad U=\sum_i U_i\,.$$

The resolution of identity $E(\lambda)$ with respect to A may be written in the form $E(U)$. When

$$E(U')\mathfrak{q}(U)=\mathfrak{q}(UU')\,,$$

put $\mathfrak{q}(U)$ instead of $\mathfrak{f}$ in (2); then we have

$$A\mathfrak{q}(U)=\int_U \lambda\mathfrak{q}(dU)\,. \qquad (3)$$

Let U be a set which is composed of only one number λ, for which $\mathfrak{q}(\lambda)$ is not a null element. In this case (3) becomes

$$A\mathfrak{q}(\lambda)=\lambda\mathfrak{q}(\lambda)\,,$$

which is nothing but (1). Thus (3) is the relation which defines the eigenelement both in the case of discrete eigenvalues and in the case of continuous eigenvalues. Hence, in the theorem of spectrum, not the points (numbers), but the sets, have important rôles. From this fact we can infer that the mathematics used in quantum mechanics must be the theory of set functions instead of the theory of point functions. The Dirac improper δ function disappears in the theory of set functions.

Ordinarily, in quantum mechanics, it is assumed that to every observable there corresponds a self-adjoint operator. And from the resolution of identity and eigenelements of that self-adjoint operator some statistical propositions are deduced. But a set of compatible measurements may be regarded as a single observation having non-numerical readings. To such a composite observation there corresponds, not one self-adjoint operator, but a set of permutable self-adjoint operators, and we have a set of permutable resolutions of identity. But this set of permutable resolutions of identity may be considered as one resolution of identity in the generalized sense. Hence the correspondence between observations and resolutions of identity is more reasonable than the correspondence between observations and self-adjoint operators.(1)

(1) G. Temple has formulated a principle of quantum theory in which projective operators are given priority over self-adjoint operators. (Cf. G. Temple [1]). But the whole discussion differs from that of my present paper.

In this paper, starting from the assumption of the probability of agreement of two states, I deduce the correspondence between observations and resolutions of identity.(1) And I prove the statistical propositions which are already known.

Ordinarily, the states being represented by point functions (wave functions), the operator corresponding to the momentum is defined by the differentiation $\frac{h}{2\pi i}\frac{d}{dq}$. But in my theory, since the states are represented by set functions, we must give a new definition to this operator. In the present paper I give a new definition which is similar to the equation of motion, and I obtain the principle of uncertainty. Lastly, from the equation of motion for the system of a free particle, I discuss the superposition of de Broglie waves, and the condition of the wave packets viewed from the basis of my theory.

Summary of Theory of Vector-Valued Set Functions.

1. Let $\mathfrak{M}$ be a multiplicative system of sets in an abstract space Ω; that is, the product of any two sets E and E' belongs to $\mathfrak{M}$ with E and E'. Assume that $\mathfrak{M}$ contains Ω itself. Let A be a set in $\mathfrak{M}$. The decomposition of A into a sum of finite or enumerably infinite disjoint sets $\{E_n\}$ belonging to $\mathfrak{M}$:

$$A=\sum_n E_n,$$

is expressed by

$$\mathfrak{D}A\equiv\sum_n E_n,$$

and the sets E_n are called the elements of the decomposition $\mathfrak{D}$. Let

$$\mathfrak{D}'A\equiv\sum_m E'_m$$

be another decomposition of A, such that E'_m is a subset of any one of the elements E_n of $\mathfrak{D}$; then we say that $\mathfrak{D}'$ is an extension of $\mathfrak{D}$. Denote by $\mathfrak{M}\mathfrak{D}A$ the aggregate of elements of all extensions of $\mathfrak{D}A$. I have said that $\mathfrak{M}\mathfrak{D}A$ is a *differential set system* in A.(2) Denote by

(1) Since the resolution of identity is a set of projections on the closed linear manifolds, my result is nothing else the relation between the calculus of propositions and the calculus of closed linear manifolds. Recently G. Birkhoff and J. v. Neumann wrote a paper concerning such a problem from the logical standpoint. (Cf. G. Birkhoff and J. v. Neumann [1], 823.)

(2) F. Maeda [5], 21.

$\mathfrak{M}A$ the aggregate of the elements of all decompositions of A. $\mathfrak{M}A$ is also a differential set system in A, and $\mathfrak{M}A$ contains A itself.

If, for any elements E of a differential set system $\mathfrak{M}\mathfrak{D}A$, one or many complex-valued function $\xi(E)$ is defined, then we say that $\xi(E)$ is a differential set function in $\mathfrak{M}\mathfrak{D}A$. When $\xi(E)$ is one-valued, and

$$\xi(E)=\sum_n \xi(E_n)$$

for any decomposition $E=\sum_n E_n$ in $\mathfrak{M}\mathfrak{D}A$, then we say that $\xi(E)$ is *completely additive.*(1)

Let $\xi(E)$ be a differential set function in $\mathfrak{M}\mathfrak{D}A$. If there exists a finite number I such that for any positive number ε, a decomposition $\mathfrak{D}_0A$ exists so that for any extension $\mathfrak{D}$ of $\mathfrak{D}_0$

$$\sup \Big|\sum_n \xi(E_n)-I\Big|<\varepsilon \qquad (\mathfrak{D}A\equiv\sum_n E_n),$$

then we say that I is the *integral* of $\xi(E)$ in A, and write

$$I=\int_A \xi(dE).$$

If $\int_A \xi(dE)$ exists, then $\int_A \xi(dE)$ exists for all elements E in $\mathfrak{M}A$, and is a completely additive differential set function in $\mathfrak{M}A$.

When $f(a)$ is a point function defined in A, then from $f(a)$ we can construct a many-valued differential set function $f(E)$ so that it takes all values $f(a)$ when a runs over all the points in E. We define the integral $\int_A f(a)\xi(dE)$ by $\int_A f(dE)\xi(dE)$.

These integrals, introduced by A. Kolmogoroff,(2) have almost all the fundamental properties of the ordinary integrals.

Let $\beta(E)$ be a completely additive, non-negative differential set function in $\mathfrak{M}\mathfrak{D}\Omega$, and $\phi(E)$ be another completely additive differential set function in $\mathfrak{M}\mathfrak{D}\Omega$. If $\phi(E)=0$ for all sets in $\mathfrak{M}\mathfrak{D}\Omega$, where $\beta(E)=0$, then we say that $\phi(E)$ is absolutely continuous with respect to $\beta(E)$.

If $\phi(E)$ is absolutely continuous with respect to $\beta(E)$, and $\int_\Omega \frac{|\phi(dE)|^2}{\beta(dE)}$ is finite, then we say that $\phi(E)$ belongs to $\mathfrak{L}_2(\beta)$. Then $\mathfrak{L}_2(\beta)$ *is a Hilbert space;* that is, (1) $\mathfrak{L}_2(\beta)$ is linear; (2) in $\mathfrak{L}_2(\beta)$, the inner pro-

(1) F. Maeda [5], 22.

(2) A. Kolmogoroff [1], 661–682.

duct $(\phi, \psi)=\int_{\Omega}\frac{\phi(dE)\overline{\psi(dE)}}{\beta(dE)}$ is defined; (3) $\mathfrak{L}_2(\beta)$ is complete.[1]

2. Let $\mathfrak{H}$ be the abstract Hilbert space, V be an abstract space, and $\mathfrak{R}\mathfrak{D}V$ be a differential set system in V. If for all sets U in $\mathfrak{R}\mathfrak{D}V$, an element $\mathfrak{q}(U)$ in $\mathfrak{H}$ be determined, and

$$\big(\mathfrak{q}(U), \mathfrak{q}(U')\big)=0 \qquad \text{when} \quad UU'=0,$$

and

$$\mathfrak{q}(U)=\sum_n (U_n)^{(2)}$$

for any decomposition $U=\sum_n U_n$ in $\mathfrak{M}\mathfrak{D}\Omega$, then we say that $\mathfrak{q}(U)$ is a *completely additive vector-valued differential set function.* In this case

$$\sigma(U)=\|\mathfrak{q}(U)\|^2$$

is a completely additive differential set function in $\mathfrak{R}\mathfrak{D}V$. And

$$\big(\mathfrak{q}(U), \mathfrak{q}(U')\big)=\sigma(UU').^{(3)} \tag{2.1}$$

Let $\xi(U)$ be a completely additive differential set function in $\mathfrak{R}\mathfrak{D}V$, which is absolutely continuous with respect to $\sigma(U)$. If there exists an element $\mathfrak{f}$ in $\mathfrak{H}$ such that for any positive number ε a decomposition $\mathfrak{D}_0 V$ exists so that for any extension $\mathfrak{D}$ of $\mathfrak{D}_0$

$$\left\|\sum_n \frac{\xi(U_n)\mathfrak{q}(U_n)}{\sigma(U_n)}-\mathfrak{f}\right\|<\varepsilon \qquad (\mathfrak{D}V\equiv\sum_n U_n),$$

then we say that $\mathfrak{f}$ is the *integral* of $\xi(U)$ by $\mathfrak{q}(U)$, and write

$$\mathfrak{f}=\int_V \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)}.$$

$\int_V \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)}$ exists when, and only when, $\xi(U)$ belongs to $\mathfrak{L}_2(\sigma)$.[4]

Since the relation (2.1) is similar to the condition of the orthogonal system $\{\mathfrak{g}_i\}$ (i being an integer)

$$(\mathfrak{g}_i, \mathfrak{g}_j)=\delta_{ij},$$

(1) F. Maeda [5], 23–31. Correction, F. Maeda [8], 107 footnote.
(2) This series is strongly convergent.
(3) F. Maeda [8], 108.
(4) F. Maeda [5], 35.

considering U as parameter, we say that $\{\mathfrak{q}(U)\}$ is an *orthogonal system of elements* in $\mathfrak{H}$.(1) When $\{\mathfrak{q}(U)\}$ is complete in $\mathfrak{H}$, any element $\mathfrak{f}$ in $\mathfrak{H}$ is expressed in the form

$$\mathfrak{f}=\int_V \frac{\xi(dU)\,\mathfrak{q}(dU)}{\sigma(dU)} \qquad \text{where} \quad \xi(U)=\big(\mathfrak{f},\,\mathfrak{q}(U)\big)\,.$$

We may say that this is the expansion of $\mathfrak{f}$ with respect to $\mathfrak{q}(U)$, and $\xi(U)$ is its expansion coefficient. Thus there is a one-to-one correspondence between $\mathfrak{H}$ and $\mathfrak{L}_2(\sigma)$. And since

$$(\mathfrak{f},\,\mathfrak{g})=(\xi,\,\eta) \tag{2.2}$$

when
$$\mathfrak{f}=\int_V \frac{\xi(dU)\,\mathfrak{q}(dU)}{\sigma(dU)}\,, \qquad \mathfrak{g}=\int_V \frac{\eta(dU)\,\mathfrak{q}(dU)}{\sigma(dU)}\,,$$

this correspondence is isomorphic.(2) We say that $\xi(U)$ is the $\mathfrak{q}$-*representative* of $\mathfrak{f}$.(3)

Let $\mathfrak{p}(E)$ be another completely additive vector-valued differential set function defined in $\mathfrak{M}\mathfrak{D}\Omega$.(4) If $\{\mathfrak{p}(E)\}$ is complete in $\mathfrak{H}$, for any element $\mathfrak{f}$ in $\mathfrak{H}$, we have the $\mathfrak{p}$-representative

$$\psi(E)=\big(\mathfrak{f},\,\mathfrak{p}(E)\big)$$

which belongs to $\mathfrak{L}_2(\beta)$, where $\beta(E)=\|\mathfrak{p}(E)\|^2$.

Put $\quad \mathfrak{U}(E,\,U)=\big(\mathfrak{q}(U),\,\mathfrak{p}(E)\big)\;$ and $\;\mathfrak{U}^*(U,\,E)=\big(\mathfrak{p}(E),\,\mathfrak{q}(U)\big)\,,$

then
$$\mathfrak{U}^*(U,\,E)=\overline{\mathfrak{U}(E,\,U)}\,,$$

$$\big(\mathfrak{U}(E,{}_{(U)}),\,\mathfrak{U}(E,{}_{(U')})\big)=\sigma(UU')\,,\text{(5)} \tag{2.3}$$

$$\big(\mathfrak{U}^*(U,{}_{(E)}),\,\mathfrak{U}^*(U,{}_{(E')})\big)=\beta(EE')\,. \tag{2.4}$$

And the relation between the $\mathfrak{q}$-representative $\xi(U)$ and the $\mathfrak{p}$-representative $\psi(E)$ of the same element $\mathfrak{f}$ in $\mathfrak{H}$ is obtained by

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(1) When V is the aggregate of integers, then $\{\mathfrak{q}(U)\}$ coincides with $\{\mathfrak{g}_i\}$.

(2) F. Maeda [2], 69–75; F. Maeda [5], 35–37; F. Maeda [8], 107–110.

(3) F. Maeda [6], 119.

(4) Ω and $\mathfrak{M}$ may or may not be the same with V and $\mathfrak{N}$.

(5) $\mathfrak{U}(E,{}_{(U)})$ means $\mathfrak{U}(E,U)$ considered as a function of set E, U being parameter.

$$\Psi(E)=\int_V \frac{\mathfrak{U}(E,dU)\xi(dU)}{\sigma(dU)} \tag{2.5}$$

and

$$\xi(U)=\int_\Omega \frac{\mathfrak{U}^*(U,dE)\Psi(dE)}{\beta(dE)}.^{(1)} \tag{2.6}$$

Hence, as Dirac says,[2] we may call $\mathfrak{U}(E,U)$ and $\mathfrak{U}^*(U,E)$ *transformation functions.*

(2.3) shows that $\{\mathfrak{U}(E,_{(U)})\}$ (U being parameter) is an orthogonal system in $\mathfrak{L}_2(\beta)$, and (2.4) shows that it is complete in $\mathfrak{L}_2(\beta)$; similarly for $\{\mathfrak{U}^*(U,_{(E)})\}$.[3] And (2.5) and (2.6) show that $\mathfrak{U}(E,U)$ and $\mathfrak{U}^*(U,E)$ are the kernels of unitary transformations between $\mathfrak{L}_2(\sigma)$ and $\mathfrak{L}_2(\beta)$.[4]

Let $\{\mathfrak{M}_U\}$ be a system of closed linear manifolds in $\mathfrak{H}$, whose index U is the set in $\mathfrak{R}V$. When $\{\mathfrak{M}_U\}$ satisfies the following conditions, then we say that $\{\mathfrak{M}_U\}$ is an *orthogonal system of closed linear manifolds* in $\mathfrak{H}$.[5]

(α) $\mathfrak{M}_U \perp \mathfrak{M}_{U'}$ when $UU'=0$,

(β) $\mathfrak{M}_U=\sum_n \mathfrak{M}_{U_n}$[6] for any decomposition $U=\sum_n U_n$ in $\mathfrak{R}V$.

In addition, if

(γ) $\mathfrak{M}_V=\mathfrak{H}$,

then we say that $\{\mathfrak{M}_U\}$ is *complete* in $\mathfrak{H}$.

When $\{\mathfrak{M}_U\}$ is a complete orthogonal system in $\mathfrak{H}$, the projecting operator $\boldsymbol{E}(U)$ on $\mathfrak{M}_U$ is called a *resolution of identity.* $\boldsymbol{E}(U)$ satisfies the following conditions:[7]

(α) $\boldsymbol{E}(U)\boldsymbol{E}(U')=\boldsymbol{E}(UU')$,

(β) $\boldsymbol{E}(U)=\sum_n \boldsymbol{E}(U_n)$ when $U=\sum_n U_n$,

(γ) $\boldsymbol{E}(V)=1$.

If $\mathfrak{q}(U)$ is a completely additive vector-valued differential set function in $\mathfrak{R}\mathfrak{D}V$, which satisfies

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(1) F. Maeda [6], 121–122.

(2) P. A. M. Dirac [2], 61 and 81.

(3) F. Maeda [1], 253.

(4) Cf. F. Maeda [3], 115–116.

(5) F. Maeda [8], 111.

(6) This means the closed linear sum.

(7) F. Maeda [8], 111–112; F. Maeda [2], 78; F. Maeda [5], 38. The ordinary defined resolution of identity $\boldsymbol{E}(\lambda)$ can be put in the above form, if we define $\boldsymbol{E}(U)=\boldsymbol{E}(\lambda_2)-\boldsymbol{E}(\lambda_1)$ when $U=[\lambda_1, \lambda_2)$.

$$E(U')\mathfrak{q}(U)=\mathfrak{q}(UU'),$$

then we say that $\mathfrak{q}(U)$ is *generated* by $E(U)$.(1)

When $\mathfrak{q}(U)$ is generated by $E(U)$, if

$$\mathfrak{f}=\int_V \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)}$$

then

$$E(U)\mathfrak{f}=\int_U \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)}.^{(2)} \tag{2.7}$$

Let $\mathfrak{q}(U)$ be generated by $E(U)$, and $\{\mathfrak{q}(U)\}$ be complete in $\mathfrak{H}$. When $\xi(U)$ is the $\mathfrak{q}$-representative of any element $\mathfrak{f}$ in $\mathfrak{H}$, then

$$E(U')\xi(U)=\xi(UU').^{(3)} \tag{2.8}$$

For, $E(U')\xi(U)$ is the $\mathfrak{q}$-representative of $E(U')\mathfrak{f}$, and it is

$$\big(E(U')\mathfrak{f},\, \mathfrak{q}(U)\big)=\big(\mathfrak{f},\, E(U')\mathfrak{q}(U)\big)=\big(\mathfrak{f},\, \mathfrak{q}(UU')\big)=\xi(UU').$$

General Assumptions in Quantum Mechanics.

3. Let us assume that any state of a dynamical system at one particular time is denoted by an element $\mathfrak{f}$ of a Hilbert space $\mathfrak{H}$, and $c\mathfrak{f}$ denotes the same state as $\mathfrak{f}$, c being any complex number. If a state $\mathfrak{f}$ may be formed by a superposition of the states $\mathfrak{f}_1$ and $\mathfrak{f}_2$, we express this relation by an equation of the type

$$\mathfrak{f}=c_1\mathfrak{f}_1+c_2\mathfrak{f}_2.$$

The different states that may be formed by the superposition of $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are given by different coefficients c_1, c_2. Let $\{\mathfrak{g}_i\}$ be a complete normalized orthogonal system in $\mathfrak{H}$; then $\mathfrak{f}$ is expressed as follows:

$$\mathfrak{f}=\sum_i c_i\mathfrak{g}_i \qquad \text{where} \quad c_i=(\mathfrak{f},\, \mathfrak{g}_i).$$

When $\mathfrak{f}$ is normalized.

$$\sum_i |c_i|^2=1.$$

Hence it is natural to assume that $|c_i|^2=|(\mathfrak{f},\, \mathfrak{g}_i)|^2$ is the probability of

(1) F. Maeda [5], 38.

(2) F. Maeda [5], 39.

(3) Here $E(U)$ is the operator in $\mathfrak{L}_2(\sigma)$, which corresponds to $E(U)$ in $\mathfrak{H}$.

agreement of $\mathfrak{f}$ and $\mathfrak{g}_i$. When $\mathfrak{g}$ is any normalized element in $\mathfrak{H}$, we take $\mathfrak{g}$ as one of the elements $\mathfrak{g}_i$ in the above-considered normalized orthogonal system. Consequently we assume that $|(\mathfrak{f}, \mathfrak{g})|^2$ *is the probability of agreement of* $\mathfrak{f}$ *and* $\mathfrak{g}$, *provided* $\mathfrak{f}$ *and* $\mathfrak{g}$ *are normalized.* The physical meaning of this assumption is as follows: Consider an observation of the state $\mathfrak{f}$ for which there is a certainty of a particular result being obtained. When this observation is made on the system in the state $\mathfrak{g}$, $|(\mathfrak{f}, \mathfrak{g})|^2$ means the probability of the same result being obtained.(1)

An observation of the dynamical system at the particular time can be described generally as a writing down of the readings from various compatible measurements, that is, the measurements which can be made simultaneously. Hence one may regard a set of compatible measurements as a single composite observation, and also admit non-numerical readings. Hence the most general form of a prediction concerning the dynamical system is that the reading determined by the observation will lie in a subset of an abstract space which we may call an *observation space.*(2)

On the basis of these assumptions I shall deduce the fundamental principles of quantum mechanics.

Statistical Propositions in Quantum Mechanics.

4. Let $\mathfrak{R}V$ be a differential set system(3) in the observation space V which corresponds to an observation. And let U be any set in $\mathfrak{R}V$. Denote by $\mathfrak{M}_U$ the aggregate of all the states $\mathfrak{f}$ of $\mathfrak{H}$ for which the reading of the observation is certainly in U. It is evident from the property of superposition that $\mathfrak{M}_U$ is a linear manifold.

Let
$$V = U + \sum_i U_i$$

be a decomposition which has U as an element. Then $\mathfrak{M}_{U_i}$ $(i=1, 2, \ldots.)$ are linear manifolds. Let $\mathfrak{f}$ be an element in $\mathfrak{H}$ which is orthogonal to

(1) For full discussions of these assumptions cf. P. A. M. Dirac [1], 18–25 and P. A. M. Dirac [2], 20–24.

(2) Cf. G. Birkhoff and J. v. Neumann [1], 824.

(3) In this section we can use a closed family (σ-Körper) instead of a differential set system. But in the following sections it is convenient to use the differential set system for the domain of definition.

all $\mathfrak{M}_{U_i}$ $(i=1, 2, \ldots.)$. Then by the assumption of sec. 3, the probability of agreement of $\mathfrak{f}$ and any element $\mathfrak{g}$ in $\mathfrak{M}_{U_i}$ $(i=1, 2, \ldots.)$ is zero. Hence the probability that the reading of the observation for $\mathfrak{f}$ is in U_i $(i=1, 2, \ldots.)$ is zero. That is, the reading of the observation for $\mathfrak{f}$ is certainly in U. Hence $\mathfrak{f}$ is in $\mathfrak{M}_U$. Consequently $\mathfrak{M}_U$ is a closed linear manifold.

By the same argument we infer that

$$\mathfrak{M}_U \perp \mathfrak{M}_{U'} \quad \text{when} \quad UU'=0\,;$$

and it is evident that

$$\mathfrak{M}_U \geqq \mathfrak{M}_{U'} \quad \text{when} \quad U \geqq U'. \tag{4.1}$$

When $U=\sum U_i$ from (4.1)

$$\mathfrak{M}_U \geqq \mathfrak{M}_{U_i} \quad \text{for all } i\,.$$

Hence
$$\mathfrak{M}_U \geqq \sum \mathfrak{M}_{U_i}\,. \tag{4.2}$$

Let $\mathfrak{f}$ be any element which is orthogonal to all $\mathfrak{M}_{U_i}$ $(i=1, 2, \ldots.)$. Then the probability that the reading of the observation for $\mathfrak{f}$ is in U_i $(i=1, 2, \ldots.)$, hence in $U=\sum U_i$, is zero. Consequently $\mathfrak{f}$ is orthogonal to $\mathfrak{M}_U$. Therefore from (4.2) it must follow that

$$\mathfrak{M}_U = \sum \mathfrak{M}_{U_i}\,.$$

Of course
$$\mathfrak{M}_V = \mathfrak{H}\,.$$

Thus, *corresponding to the observation, we have a complete orthogonal system of closed linear manifolds* $\{\mathfrak{M}_U\}$ *in* $\mathfrak{H}$,(1) *whose index is the set* U *in the observation space* V. I say that $\{\mathfrak{M}_U\}$ is a *spectral decomposition* of $\mathfrak{H}$ with respect to the observation.

Let $\boldsymbol{E}(U)$ be a projecting operator on $\mathfrak{M}_U$; then $\boldsymbol{E}(U)$ is a resolution of identity. Thus, *corresponding to the observation we have a resolution of identity* $\boldsymbol{E}(U)$ *whose index is the set* U *of the observation space* V.

Let O_1 and O_2 be two observations with the observation spaces V_1 and V_2 respectively, and $\boldsymbol{E}_1(U_1)$, $\boldsymbol{E}_2(U_2)$ be the resolutions of identity which correspond to O_1 and O_2. When $\boldsymbol{E}_1(U_1)$ and $\boldsymbol{E}_2(U_2)$ are permuta-

(1) Cf. sec. 2.

ble, then $E_1(U_1)E_2(U_2)$ $\left(=E(U_1, U_2) \text{ say}\right)$ is a resolution of identity defined in the composite space (V_1, V_2),(1) in which the product of (U_1, U_2) and (U_1', U_2') is (U_1U_1', U_2U_2'). Thus we have a single observation with observation space (V_1, V_2). Hence the two observations O_1 and O_2 can be made simultaneously. That is, O_1 and O_2 are compatible.

Conversely, when O_1 and O_2 are compatible, we may regard them as a single composite observation (O_1, O_2) with observation space (V_1, V_2). Then there exists the spectral decomposition $\{\mathfrak{M}_{(U_1, U_2)}\}$ of $\mathfrak{H}$ with respect to (O_1, O_2). From the definition, $\{\mathfrak{M}_{(U_1, V_2)}\}$ is the spectral decomposition of $\mathfrak{H}$ with respect to O_1. And similarly for $\{\mathfrak{M}_{(V_1, U_2)}\}$. Hence, if we denote by $E(U_1, U_2)$ the resolution of identity which corresponds to (O_1, O_2), then it must follow

that $$E_1(U_1)=E(U_1, V_2), \qquad E_2(U_2)=E(V_1, U_2).$$

Hence $$E_1(U_1)E_2(U_2)=E(U_1, U_2)=E_2(U_2)E_1(U_1).$$

Thus the two resolutions of identity $E_1(U_1)$ and $E_2(U_2)$ are permutable.

Consequently, *a necessary and sufficient condition that two observations O_1 and O_2 are compatible, is that the corresponding resolutions of identity $E_1(U_1)$ and $E_2(U_2)$ are permutable.*

Let $E(U)$ be a resolution of identity which corresponds to an observation, and let a state be denoted by a normalized $\mathfrak{f}$. When $V=U+\sum_i U_i$, then

$$\mathfrak{f}=E(U)\mathfrak{f}+\sum_i E(U_i)\mathfrak{f}.$$

Since $E(U)\mathfrak{f}$ and $E(U_i)\mathfrak{f}$ belong to $\mathfrak{M}_U$ and $\mathfrak{M}_{U_i}$ respectively, the probability that the reading of the observation is in U is 1 for the state $E(U)\mathfrak{f}$, and is 0 for the state $E(U_i)\mathfrak{f}$. But by sec. 3, the probability of agreement of $\mathfrak{f}$ and $E(U)\mathfrak{f}$ is $\left|\left(\mathfrak{f}, \frac{E(U)\mathfrak{f}}{\|E(U)\mathfrak{f}\|}\right)\right|^2=\|E(U)\mathfrak{f}\|^2$. Hence we have the following theorem:

The probability that the reading of the observation for the state $\mathfrak{f}$ is in U is $\|E(U)\mathfrak{f}\|^2$, provided $\mathfrak{f}$ is normalized.(2)

This proposition satisfies the properties of probability. First, the addition theorem of probability holds. For when $U=\sum_i U_i$, $\|E(U)\mathfrak{f}\|^2=$

(1) For, $E_1(U_1)E_2(U_2)$ is a projection, and satisfies the conditions (α), (β) and (γ) in sec. 2.

(2) J. v. Neumann [1], 104.

$\sum_i \| E(U_i)\mathfrak{f}\|^2$. And secondly, the probability that the reading of the observation is in V is 1, for $\| E(V)\mathfrak{f}\|^2=\|\mathfrak{f}\|^2=1$.

5. For a given resolution of identity $E(U)$ which corresponds to an observation, we have a complete orthogonal system of elements in $\mathfrak{H}$, which is generated by $E(U)$.(1) For simplicity's sake consider the case where this system is composed of only one vector-valued set function $\mathfrak{q}(U)$.(2) Let $\mathfrak{f}$ be any state. Then

$$\xi(U)=\big(\mathfrak{f},\ \mathfrak{q}(U)\big)$$

is the representative of $\mathfrak{f}$. And we have

$$\mathfrak{f}=\int_V \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)}.^{(3)} \qquad (5.1)$$

Then since
$$E(U)\mathfrak{f}=\int_U \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)},$$

we have by (2.2)
$$\| E(U)\mathfrak{f}\|^2=\int_U \frac{|\xi(dU)|^2}{\sigma(dU)}.$$

Thus *the probability that the reading of the observation for the state* $\mathfrak{f}$ *is in* U *is* $\int_U \frac{|\xi(dU)|^2}{\sigma(dU)}$, *provided* $\mathfrak{f}$ *is normalized.*(4)

The proposition above may be called the generalized form of the *principle of interference.* For, when the observation is the measurement of position, U is a set in the *xyz*-space, and $\xi(U)$ corresponds to the so-called wave function.(5)

But from (5.1), $\xi(U)$ is the coefficient of expansion of $\mathfrak{f}$ with respect to $\{\mathfrak{q}(U)\}$ which is generated by $E(U)$. Hence the proposition given above is nothing but the *principle of spectral decomposition* in generalized form.(6) Thus, in general theory, these two fundamental principles in wave mechanics come together.

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(1) F. Maeda [2], 80; [8], 114.

(2) For the general case, cf. F. Maeda [6], 123–127, 131–132.

(3) Cf. sec. 2.

(4) Cf. F. Maeda [6], 129. This is the complete form of the theorem obtained by Dirac. Cf. P. A. M. Dirac [2], 65, 78.

(5) L. de Broglie [2], 22.

(6) L. de Broglie [2], 165.

Especially when the given observation has a numerical reading, the observation space V is the space of real numbers. If $E(U)$ is the resolution of identity corresponding to the observation, then

$$A\mathfrak{f}=\int_V \lambda E(dU)\mathfrak{f}$$

is a self-adjoint operator. In this case we may say that *the observation is represented by the self-adjoint operator* A. Let $\mathfrak{f}$ be a state which is in the domain of A. When $\mathfrak{f}$ is normalized, $\|E(U)\mathfrak{f}\|^2$ is the probability that the reading of the observation A for the state $\mathfrak{f}$ is in U. Hence $\int_V \lambda \|E(dU)\mathfrak{f}\|^2$ is the average value of A for the state $\mathfrak{f}$. Since

$$\int_V \lambda \|E(dU)\mathfrak{f}\|^2=\int_V \lambda\big(E(dU)\mathfrak{f},\mathfrak{f}\big)=(A\mathfrak{f},\mathfrak{f}),$$

we have the following theorem:

If the observation, represented by a self-adjoint operator A, for the system represented by the element $\mathfrak{f}$, is made a large number of times, the average of all the results obtained will be $(A\mathfrak{f},\mathfrak{f})$, provided $\mathfrak{f}$ is normalized.(1)

q- and p-Observations.

6. Let us take a dynamical system described by a set of canonical coordinates and momenta q_r, p_r $(r=1, 2, \ldots, n)$. Then we have two observations, which we may call q- and p-observations. The observation space of q-observation is the n-dimensional euclidean space, say $R_n^{(q)}$, which has $q_1, q_2, \ldots, q_n$ coordinates. And the observation space of p-observation is also the n-dimensional euclidean space, say $R_n^{(p)}$, which has $p_1, p_2, \ldots, p_n$ coordinates. Then the q-representative of a state in $\mathfrak{H}$ is the so-called wave function $\psi(E)=\psi(E_1, E_2, \ldots, E_n)$(2) which belongs to $\mathfrak{L}_2(\beta)$, where $\beta(E)=\beta(E_1, E_2, \ldots, E_n)$ is the volume of the interval $(E_1, E_2, \ldots, E_n)$ in $R_n^{(q)}$.(3) And the probability that the reading of the

(1) P. A. M. Dirac starts from this proposition in his quantum mechanical description. (P. A. M. Dirac [2], 43).

(2) Ordinarily, the wave function is expressed by a point function $f(q_1, q_2, \ldots, q_n)$. Here we use the set function $\psi(E)=\int_E f(q_1, q_2, \ldots, q_n)\,\beta(dE)$.

(3) Here the differential set system is composed of the finite intervals in $R_n^{(q)}$.

q-observation for the state $\psi(E)$ is in E is $\int_E \frac{|\psi(dE)|^2}{\beta(dE)}$, provided $\psi(E)$ is normalized.(1) Hence, by (2.8), *the resolution of identity $E_q(E)$, which corresponds to the q-observation, must be defined by the relation*

$$E_q(E')\psi(E)=\psi(EE').^{(2)} \tag{6.1}$$

Denote the q_i-space by $R_1^{(q_i)}$. Then $R_n^{(q)}=(R_1^{(q_1)}, R_1^{(q_2)}, \ldots, R_1^{(q_n)})$. Put

$$E_{q_i}(E_i)=E_q(R_1^{(q_1)}, R_2^{(q_2)}, \ldots, E_i, \ldots, R_1^{(q_n)}).$$

Then $E_{q_i}(E_i)$ is also a resolution of identity, which has the following property

$$E_{q_i}(E_i')\psi(E_1, E_2, \ldots, E_i, \ldots, E_n)=\psi(E_1, E_2, \ldots, E_iE_i', \ldots, E_n).$$

We may say that the self-adjoint operator

$$Q_i\psi=\int_{R_1^{(q_i)}} q_iE_{q_i}(dE_i)\psi$$

represents the observation q_i.

To obtain the operator which represents p_i, consider a translation in $R_1^{(q_i)}$, such that

$$q_i'=q_i+s_i.$$

And denote the translated set of E_i by $T_{s_i}E_i$. Put

$$\begin{aligned}\psi_{s_i}(E_1, E_2, \ldots, E_i, \ldots, E_n)&=\psi(E_1, E_2, \ldots, T_{s_i}E_i, \ldots, E_n)\\ &=U_{s_i}\psi(E_1, E_2, \ldots, E_i, \ldots, E_n).\end{aligned}$$

Then U_{s_i} is a unitary operator with group property and continuity property.(3) Then there exists a unique resolution of identity $E_{p_i}(U_i)$ defined in the space of real numbers $R_1^{(p_i)}$ such that

$$U_{s_i}\psi=\int_{R_1^{(p_i)}} e^{\frac{2\pi i}{h}s_ip_i}E_{p_i}(dU_i)\psi. \tag{6.2}$$

And if we put $$P_i\psi=\int_{R_1^{(p_i)}} p_iE_{p_i}(dU_i)\psi,$$

(1) This is the so-called principle of interference.
(2) Cf. F. Maeda [5], 42–43.
(3) Cf. F. Maeda [4], 8–9.

then $$\underset{\Delta s_i\to 0}{[\lim]}\frac{U_{\Delta s_i}\Psi_{s_i}-\Psi_{s_i}}{\Delta s_i}=\frac{2\pi i}{h}P_i\Psi_{s_i}$$

provided Ψ_{s_i} is in the domain of P_i.(1) We may write this

$$\frac{\partial\Psi_{s_i}}{\partial s_i}=\frac{2\pi i}{h}P_i\Psi_{s_i}.^{(2)} \tag{6.3}$$

Thus we have a resolution of identity which corresponds to p_i. When $U=(U_1, U_2, \ldots\ldots, U_n)$ is a set in $R_n^{(p)}=(R_1^{(p_1)}, R_1^{(p_2)}, \ldots\ldots, R_1^{(p_n)})$, put

$$E_p(U)=E_{p_1}(U_1)E_{p_2}(U_2)\ldots\ldots E_{p_n}(U_n).$$

Then $E_p(U)$ is also a resolution of identity. We may say that $E_p(U)$ is the resolution of identity which corresponds to the p-observation.

Put $$\mathfrak{U}_0(E, U)=\frac{1}{h^{\frac{n}{2}}}\int_E\beta(dE)\int_U e^{\frac{2\pi i}{h}(q_1p_1+q_2p_2+\ldots\ldots+q_np_n)}\beta(dU). \tag{6.4}$$

Then $$\int_{R_n^{(p)}}\frac{\mathfrak{U}_0(E, dU)\overline{\mathfrak{U}_0(E', dU)}}{\beta(dU)}=\beta(EE'),^{(3)}$$

and $$\int_{R_n^{(q)}}\frac{\mathfrak{U}_0(dE, U)\overline{\mathfrak{U}_0(dE, U')}}{\beta(dE)}=\beta(UU').$$

Hence $\{\mathfrak{U}_0(E, {}_{(U)})\}$ is a complete orthogonal system in the q-representation space $\mathfrak{L}_2(\beta)$.(4)

Let $F(U)$ be a resolution of identity which generates $\{\mathfrak{U}_0(E, {}_{(U)})\}$.(5)

And put $F_i(U_i)=F(R_1^{(p_1)}, R_1^{(p_2)}, \ldots\ldots, U_i, \ldots\ldots, R_1^{(p_n)})$.

Then, since

$$F_i(U_i')\mathfrak{U}_0(E, U_1, \ldots\ldots, U_i, \ldots\ldots, U_n)=\mathfrak{U}_0(E, U_1, \ldots\ldots, U_1U_1', \ldots\ldots, U_n),$$

(1) Cf. J. v. Neumann [2], 569–573; and M. H. Stone [1], 644. [lim] means the strong convergence of the limit.

(2) Thus we have a similar equation to the equation of motion (8.1).

(3) This is obtained by the relation

$$\frac{1}{h}\int_{-\infty}^{\infty}dp\int_a^b e^{\frac{2\pi i}{h}qp}dq\int_c^d e^{-\frac{2\pi i}{h}q'p}dq'=\text{length common to } (a, b) \text{ and } (c, d).$$

(4) Cf. sec. 2.

(5) For the construction of such a resolution of identity, cf. F. Maeda [2], 81; [8], 114.

we have
$$\int_{R_1^{(p_i)}} e^{\frac{2\pi i}{h}s_i p_i} F_i(dU_i)\,\mathfrak{U}_0(E, U)$$
$$=\frac{1}{h^{\frac{n}{2}}}\int_U \beta(dU)\int_E e^{\frac{2\pi i}{h}\{q_1p_1+\dots+(q_i+s_i)p_i+\dots+q_np_n\}}\,\beta(dE)\,.$$

That is,
$$U_{s_i}\mathfrak{U}_0(E,{}_{(U)})=\int_{R_1^{(p_i)}} e^{\frac{2\pi i}{h}s_i p_i} F_i(dU_i)\,\mathfrak{U}_0(E,{}_{(U)})\,.$$

The system $\{\mathfrak{U}_0(E,{}_{(U)})\}$ being dense in $\mathfrak{L}_2(\beta)$, from the continuity property of U_{s_i} we have
$$U_{s_i}\psi(E)=\int_{R_1^{(p_i)}} e^{\frac{2\pi i}{h}s_i p_i} F_i(dU_i)\,\psi(E)$$
for any $\psi(E)$ in $\mathfrak{L}_2(\beta)$. Hence from (6.2) it must follow that
$$F_i(U_i)=E_{p_i}(U_i)$$
for all $i=1, 2, \dots, n$.

Thus *the resolution of identity* $E_p(U)$ *which corresponds to the p-observation is that which generates* $\{\mathfrak{U}_0(E,{}_{(U)})\}$.

Since by (6.1) $\{\beta(EE')\}$ (E' being parameter) is a complete orthogonal system generated by $E_q(E)$,
$$\big(\mathfrak{U}_0(E, U),\, \beta(EE')\big)=\mathfrak{U}_0(E', U)$$
is the transformation function between q-representation and p-representation. And by (2.6) the p-representative $\xi(U)$ of $\psi(E)$ is
$$\xi(U)=\int_{R_n^{(q)}}\frac{\overline{\mathfrak{U}_0(dE, U)}\,\psi(dE)}{\beta(dE)}\,. \tag{6.5}$$

Principle of Uncertainty.

7. Now we obtain the principle of uncertainty, as follows: Let $\psi(E)$ represent any state for which the reading of the q-observation is certainly in E'. Then, $\psi(E)$ being normalized, we have
$$\int_{E'}\frac{|\psi(dE)|^2}{\beta(dE)}=1=\int_{R_n^{(q)}}\frac{|\psi(dE)|^2}{\beta(dE)}\,. \tag{7.1}$$

Hence $\psi(E)$ must vanish for any interval E outside E'. From (6.5) the p-representative of $\psi(E)$ is

$$\xi(U)=\int_{E'}\frac{\overline{\mathfrak{U}_0(dE,\,U)}\,\psi(dE)}{\beta(dE)}\,.$$

Hence
$$|\xi(U)|^2\leqq\int_{E'}\frac{|\mathfrak{U}_0(dE,\,U)|^2}{\beta(dE)}\int_{E'}\frac{|\psi(dE)|^2}{\beta(dE)}\,. \tag{7.2}$$

But by (6.4)
$$|\mathfrak{U}_0(E,\,U)|^2\leqq\frac{1}{h^n}[\beta(E)\beta(U)]^2\,,$$

hence
$$\int_{E'}\frac{|\mathfrak{U}_0(dE,\,U)|^2}{\beta(dE)}\leqq\frac{1}{h^n}\beta(E')[\beta(U)]^2\,.^{(1)}$$

Therefore, from (7.1) and (7.2), we have

$$|\xi(U)|^2\leqq\frac{1}{h^n}\beta(E')[\beta(U)]^2\,.$$

Consequently we have

$$\int_U\frac{|\xi(dU)|^2}{\beta(dU)}\leqq\frac{1}{h^n}\beta(E')\beta(U)\,.^{(1)}$$

Thus, taking $\varDelta q=\beta(E')$, $\varDelta p=\beta(U)$, we obtain the following principle of uncertainty:

Take any state for which the reading of the q-observation is certainly in the range of width $\varDelta q$. Then the probability that the reading of the p-observation for that state, is in the range of width $\varDelta p$ is not greater than $\dfrac{\varDelta q\varDelta p}{h^n}$.(2)

If we take, instead of $\psi(E)$, the p-representative of the state, we obtain the same result with q and p interchanged.

Equations of Motion.

8. In the preceding section I have given the general scheme of relations between states and observations at one instant of time. Now we shall consider the connexion between different instants of time. As

(1) By the definition of the integral in sec. 1. Cf. F. Maeda [5], 24.
(2) This is the extension obtained in F. Maeda [6], 136.

in my previous paper,[1] I argue on the assumption that the general form of the equation of motion is

$$\mathfrak{f}_{t_2}=U_{(t_1,t_2)}\mathfrak{f}_{t_1},$$

where $\mathfrak{f}_t$ represents the state at time t, and $U_{(t_1,t_2)}$ is a unitary operator which depends on the time interval (t_1, t_2). When $U_{(t_1,t_2)}$ depends only on the length of the time interval, and not on the position of the time interval, there exists a self-adjoint operator H so that

$$[\lim]_{\Delta t\to 0}\frac{\mathfrak{f}_{t+\Delta t}-\mathfrak{f}_t}{\Delta t}=-\frac{2\pi i}{h}H\mathfrak{f}_t$$

when $\mathfrak{f}_t$ is in the domain of H.[2] This may be written as

$$\frac{\partial\mathfrak{f}_t}{\partial t}=-\frac{2\pi i}{h}H\mathfrak{f}_t. \tag{8.1}$$

Thus we have Schrödinger's form of the equation of motion with constant Hamiltonian H. The solution of (8.1) is

$$\mathfrak{f}_t=\int_{R_1}e^{-\frac{2\pi i}{h}Ht}E_H(dU)\mathfrak{f}, \tag{8.2}$$

where $E_H(U)$ is the resolution of identity which corresponds to H. We assume that the Hamiltonian H is of the same form as that in classical mechanics.[3]

In what follows, we especially consider the system consisting of a free particle. In this case, the Hamiltonian in classical mechanics is

$$H=\frac{1}{2m}(p_x^2+p_y^2+p_z^2),$$

where m is the mass of the particle. Hence in quantum mechanics we put

$$H=\frac{1}{2m}(P_x^2+P_y^2+P_z^2).$$

Let the q-representative of the state of the system at time t be $\psi_t(E_1, E_2, E_3)$. And put, as in sec. 6,

(1) F. Maeda [7], 287–288.
(2) By the same argument as in sec. 6.
(3) For the constants of motion cf. F. Maeda [7], 289–290.

$$\Psi_{x,y,z,t}(E_1, E_2, E_3) = \Psi_t(T_xE_1, T_yE_2, T_zE_3)\,.$$

Then, from (6.3) $\quad \dfrac{\partial \Psi_{x,y,z,t}}{\partial x} = \dfrac{2\pi i}{h} P_x \Psi_{x,y,z,t}\,.$

Similarly for y and z. Hence the equation of motion (8.1) becomes

$$\frac{4\pi i}{h} m \frac{\partial \Psi}{\partial t} + \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} = 0\,,$$

where Ψ stands for $\Psi_{x,y,z,t}(E_1, E_2, E_3)$, and the operators $\dfrac{\partial}{\partial t}$, $\dfrac{\partial}{\partial x}$, $\dfrac{\partial}{\partial y}$, $\dfrac{\partial}{\partial z}$ have the meanings specified above.

In this particular case, (8.2) becomes

$$\mathfrak{f}_t = \int_{R_3^{(p)}} e^{-\frac{2\pi i}{h}Ht} E_H(dU)\,\mathfrak{f}\,, \tag{8.3}$$

where $H = \dfrac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$ and $E_H(U) = E_{p_x}(U_1) E_{p_y}(U_2) E_{p_z}(U_3)$, U being (U_1, U_2, U_3). When $\mathfrak{f}$ is represented by

$$\mathfrak{U}_0(E, {}_{(U)}) = \frac{1}{h^{\frac{3}{2}}} \int_E \beta(dE) \int_U e^{\frac{2\pi i}{h}(p_x x + p_y y + p_z z)} \beta(dU)\,, \tag{8.4}$$

then, since $\quad E_H(U')\,\mathfrak{U}_0(E, U) = \mathfrak{U}_0(E, UU')\,,$

we have

$$\mathfrak{U}_{0,t}(E, {}_{(U)}) = \frac{1}{h^{\frac{3}{2}}} \int_E \beta(dE) \int_U e^{\frac{2\pi i}{h}(p_x x + p_y y + p_z z - Ht)} \beta(dU)\,, \tag{8.5}$$

where $\quad H = \dfrac{1}{2m}(p_x^2 + p_y^2 + p_z^2)\,.$

When U is a very small region with centre (p_x, p_y, p_z), then (8.5) may be considered as a wave with frequency $\nu = \dfrac{H}{h}$, moving in the direction specified by the vector (p_x, p_y, p_z). Hence it corresponds to de Broglie waves. Thus, strictly speaking, (8.5) is a group of waves, which is formed by superposing a number of de Broglie waves belonging to different values of (p_x, p_y, p_z) in U.

But the general form of a group of waves is

$$\Psi_t(E) = \frac{1}{h^{\frac{3}{2}}} \int_{R_3^{(p)}} a(p_x, p_y, p_z)\, \beta(dU) \int_E e^{\frac{2\pi i}{h}(p_x x + p_y y + p_z z - Ht)} \beta(dE)\,, \tag{8.6}$$

where $a(p_x, p_y, p_z)$ vanishes except in a region U_0. For, at $t=0$, in spite of (8.4), we are considering a superposition of de Broglie waves

$$\psi(E)=\int_{R_3^{(p)}}\frac{\xi(dU)\,\mathfrak{U}_0(E,dU)}{\beta(dU)} \tag{8.7}$$

where $\xi(U)=\int_U a(p_x, p_y, p_z)\,\beta(dU)$. Since $\mathfrak{U}_0(E,(U))$ is generated by $E_H(U)$, by (2.7)

$$E_H(U)\,\psi(E)=\int_U\frac{\xi(dU)\,\mathfrak{U}_0(E,dU)}{\beta(dU)}.$$

Hence
$$\psi_t(E)=\int_{R_3^{(p)}}e^{-\frac{2\pi i}{h}Ht}E_H(dU)\,\psi(E)$$

$$=\int_{R_3^{(p)}}\frac{e^{-\frac{2\pi i}{h}Ht}\xi(dU)\,\mathfrak{U}_0(E,dU)}{\beta(dU)}. \tag{8.8}$$

This the same as (8.6).

By (8.7) $\xi(U)$ is the p-representative of $\psi(E)$. Hence

$$E_{p_x}(U_1')\,\xi(U_1, U_2, U_3)=\xi(U_1U_1', U_2, U_3).$$

Similarly for $E_{p_y}(U_2')$ and $E_{p_z}(U_3')$. Consequently

$$E_H(U')\,\xi(U)=\xi(UU').$$

Therefore, by (8.3),
$$\xi_t(U)=\int_U e^{-\frac{2\pi i}{h}Ht}\xi(dU). \tag{8.9}$$

This is the solution of (8.1) in the p-representation. Then (8.8) becomes

$$\psi_t(E)=\int_{R_3^{(p)}}\frac{\xi_t(dU)\,\mathfrak{U}_0(E,dU)}{\beta(dU)}. \tag{8.10}$$

This is obvious, since $\mathfrak{U}_0(E, U)$ is the transformation function between q- and p-representations.

Now we can find the condition under which a wave packet is represented by a group of waves. Let us assume that the group of waves $\psi_t(E)$ in (8.10) is also a wave packet such that it vanishes identically outside a certain region E_0. From (8.9), $\xi_t(U)$ vanishes identically outside a region U_0. But (8.10) shows that $\psi_t(E)$ and $\xi_t(U)$ are the q- and the p-representative of the same state. Since at this state the reading of the q-observation is certainly in E_0, and the probability that the reading of the p-observation is in U_0 is 1, from the principle of uncertainty in sec. 7 we have

$$\frac{\Delta q \Delta p}{h^3} \geqq 1,$$

where $\Delta q = \beta(E_0)$, $\Delta p = \beta(U_0)$. Thus we arrive at the conclusion:

When a wave packet of width Δq is represented by a group of waves of width Δp, it must be that

$$\Delta q \Delta p \geqq h^3.$$

This is another form of the principle of uncertainty.(1)

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(1) Cf. L. de Broglie [1], 49–58; [2], 23–27.

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